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XXVI. *A Comparison of the Standard Resistance-coils of the British Association with Mercury Standards constructed by* M. J. R. Benoit *of Paris and* Herr Strecker *of Würzburg. By* R. T. GLAZEBROOK, *M.A., F.R.S., Fellow and Assistant Tutor of Trinity College, Cambridge**.

IN accordance with the resolution of the Electrical Congress held in Paris in 1884, the Electrical-Standards Committee of the British Association decided to have made a series of resistance-coils in terms of the Legal Ohm; and the work of testing and comparing these coils was entrusted to me. In Paris, at the request of the Minister for the Post Office and Telegraphs, M. J. R. Benoit undertook the same task. An account of his experiments is published in the *Journal de Physique* for January 1885.

M. Benoit started *ab initio*, and constructed a series of glass tubes, the electrical resistance of which, when filled with mercury, can be calculated from their dimensions. Four of these were made, each having a resistance of about 1 ohm.

* Read May 23, 1885.

The plan adopted by the Committee was different. The specific resistance of mercury in terms of the British-Association unit has been carefully determined by Lord Rayleigh and others.

For the purpose of constructing Legal Ohms, it was decided to adopt a number for this quantity founded on their experiments, and then work from the British-Association standards. To adopt the other course would only have been to repeat at some trouble the experiments of Lord Rayleigh, M. Mascart, Herr Strecker, and others.

M. Benoit made a number of copies of his mercury-standards, and after reading his paper, I wrote and asked him to send me one or more of these, that I might compare them directly with the coils which had by that time been constructed for me by Messrs. Elliott Bros. M. Benoit replied most courteously, and sent me, about Easter, three of his copies. The object of the present paper is to give an account of the comparison of these with the standards belonging to the British Association.

M. Benoit's copies were mercury-standards. Each consisted of a glass tube bent several times, A A, fig. 1. The ends of this tube are ground flat and pass into two glass cups, B, B, which for most of their length are considerably wider than the tube, but taper down at their lower ends. The tubes are connected to the cups by short bits of india-rubber tubing, C, C, which fit tightly over the narrow ends of the cups, and are secured by wrapping string firmly over the india-rubber. To make the joint tight the india-rubber and glass were varnished over with shellac varnish. The cups open at the top, and can be closed by glass stoppers.

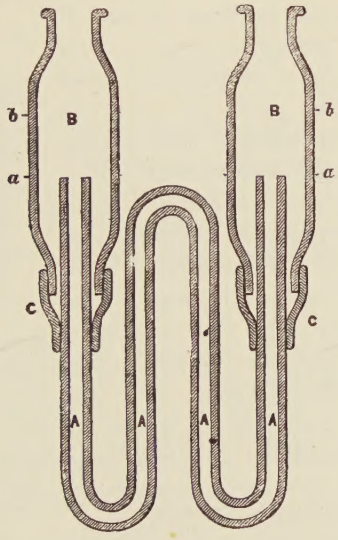
The standards were accompanied by full directions for use. I give a free translation of the greater part of the instructions.

1. On each of the cups are engraved two marks *a a*, *b b*. The lower mark *a a* indicates the position which the extremity of the tube introduced into the cup ought to occupy. The upper mark *b b* indicates the top of the mercury in the tube when filled. These two conditions must be fulfilled approximately; an error of ·5 centim. in their sum changes the resistance either way by about ·00018 ohm.

2. *Method of Filling.*—The tube may be filled in air in the

following manner:—Place it nearly horizontally and pour some mercury into one of the cups; then incline it gradually, shaking it a little, until the mercury enters the tube. The mercury should fill the tube slowly, without leaving a trace of air. Add mercury always by the same cup until it reaches the upper mark *b b* in both cups. The method is not always successful; it is better to work in a vacuum.

Fig. 1.

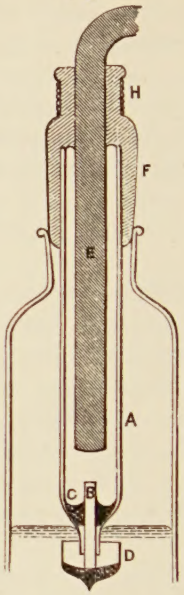


[Various methods for doing this are possible; it is perhaps hardly necessary to give the details of the one employed by M. Benoit.]

3. *Method of Using the Standard.*—The standard is introduced into the circuit by means of contact-pieces dipping into the mercury in the cups. These pieces should not reach more than a few millimetres below the surface. Contact-pieces of amalgamated copper cannot be used, because they render the mercury impure and alter its resistance rapidly, diminishing it appreciably in a few hours. Platinum, again, does not make a constant contact with the mercury; and the uncertainty produced by its variability is fatal to its use. To avoid this double difficulty, M. Benoit employed contact-pieces of a

special form (fig. 2). A glass tube A is drawn out to a point at its lower end; through this point a platinum wire B passes. The wire is held in position by shellac C; and a small cup D of thin glass is attached to its lower end also by shellac. Thus the two separate glass portions A and D are in electrical communication by means of the platinum wire B. The two portions A and D are filled with mercury in a vacuum. The mercury is thus brought into good contact with the platinum, and the mercury-platinum contact remains unchanged during the observations. Into the upper division A one end of a stout copper rod E is plunged; the other end of this rod forms the connection with the rest of the circuit. The cup D dips a few millimetres below the mercury in the standard. The mercury which fills it remains constantly in contact with the platinum wire, and may be kept without change almost indefinitely. These contact-pieces introduce an appreciable resistance of 2 to 3 thousandths of an ohm into the circuit; but this resistance remains constant, and can be eliminated by a method of substitution.

Fig. 2.



The standard is supported in a glass vessel, and can be reduced in temperature to zero by filling the vessel with finely-broken ice. The values of the standards are given at zero. The value at any other temperature is given by the formula

$$R_t = R_0(1 + \cdot 0008649t + \cdot 00000112t^2),$$

t being the temperature in degrees Centigrade.

4. *Preparation of the Mercury.*—The mercury may be purified by the action of nitric acid, and dried by means of sulphuric acid and caustic potash.

Experiment showed that mercury from different sources, even when it had been rendered impure by the admixture of copper, lead, or zinc, after being treated by this process, gave

the same results. The process, however, does not free the mercury from silver or from the less oxidizable metals.

[In my experiments described below the mercury was freshly distilled by the aid of the admirable piece of apparatus designed by Weinhold, and introduced into England by Mr. W. N. Shaw.]

5. *Method of Cleaning the Tubes.*—The tubes should be cleaned by passing through them in succession:—

1. Distilled water.
2. Strong nitric acid.
3. Distilled water.
4. Ammonia.
5. Distilled water.

They must then be dried by a current of dry air.

To clean the tubes it is best to separate them from the cups. To effect this, the pieces of india-rubber tubing which connect them should be cut. To make the glass slip into the india-rubber tubing, it should be moistened with a drop of benzine.

In preparing the tubes for my measurements, I endeavoured to carry out the above instructions as carefully as possible. The resistance comparisons were made by Carey Foster's method, using the wire bridge of the British Association as designed by Dr. Fleming.

In order to compare the resistance of one of the tubes with a standard coil, it was necessary either to know accurately or to eliminate the resistance of the contact-pieces. It was also important to determine within what limits the resistance of the contact-pieces remained constant, and whether they could be repeatedly filled with mercury in such a way as to retain the same resistance.

For this purpose they were compared with a short piece of copper rod. The cups D (fig. 2), one belonging to each contact-piece, were immersed in the same beaker of mercury, being placed close together and covered to the same depth as when actually in use; the other ends of the copper rods E connected in the usual way to the bridge. The contact-tubes had been filled by being placed in a large test-tube which was exhausted, and into which mercury was then

admitted by means of a tap attached to the test-tube. The resistance of the contact-pieces was thus found in terms of the divisions of the bridge-wire before and after a set of comparisons of the mercury-tubes and standard coils.

It was found that the resistance of the contact pieces remained very nearly constant during the time occupied in making a series of comparisons. The variations in the values found before and after rarely amounted to more than 1 bridge-wire division, or about ·00005 ohm; and the mean of the two values obtained is pretty certainly correct to less than this.

At the same time different fillings of the same contact-tubes led to very different values for their resistances. This was no doubt due to small irregularities in the contact between the platinum and the mercury. Thus I found the following values in terms of a bridge-wire division :—

Date &c.	Resistance of contacts.
March 25	93·4 b.w.d.
„ Contacts refilled	69·4 „
„ 26 { Copper rods being left in the mercury overnight }	68·2 „
„ „ Contacts refilled, one renewed	114·5 „
„ 28 Copper rods being left in since 26th...	109·4 „

The differences between the 69·4 and the 68·2, and, again, between the 114·5 and 109·4, are due, I believe, to the copper of the contact-rods being dissolved by the mercury in the tubes.

On March 28th, the contact-tubes were cleaned and refilled, and the resistance was found to be 57·2 b.w.d. Throughout the experiments until April 9, when one of them was broken, their values did not differ greatly from the above.

On April 11th, new contact-tubes were prepared and used throughout the rest of the experiments; their value remained fairly constant at about 58 b.w.d.

Thus the quantity actually determined by the observations was the difference between the resistance of the mercury tube + the contact-pieces and the standard coil. In order to get

the resistance of the mercury tube in terms of the standard coil, the resistance contact-pieces found as above described had to be subtracted.

Another way of eliminating the resistance of the contacts from the result is to compare first the mercury tube and then the standard with some other coil, using the same contact-pieces to connect the mercury tube and the standard to the bridge. This can be done by removing the contact-tubes from the mercury tube, and dipping them into two mercury cups, into which the electrodes of the standard also dip. Since, however, my apparatus allowed me to measure the resistance of the contact-tubes with all the accuracy I desired, the first method was the more convenient, as it necessitated fewer observations.

The comparisons were made both with the original standards of the British Association—B.A. Units—and also with the Legal-Ohm Standards of the same, recently constructed for me by Messrs. Elliott Bros., taking as a basis the value for the resistance of mercury in terms of the B.A. unit adopted by the British-Association Committee. According to this,

$$1 \text{ Legal Ohm} = 1{\cdot}0112 \text{ B.A.U.}$$

The three mercury tubes Nos. 37, 38, 39, sent by M. Benoit, were compared. Between two observations of resistance the mercury was occasionally drawn through the tubes by connecting the cups alternately to the air-pump. I hoped in this manner to displace any very small air-bubble or particle of dust which might have got lodged in the tube. In the final set of observations recorded below, it will be seen that no appreciable change in the resistance was detected by this precaution. I therefore suppose the tubes to have been properly filled.

In some earlier observations, which will be referred to again shortly, variations of considerable amount were produced by passing the mercury through. In the tables below the values of the resistances which were found by comparison with the B.A.U., and reduced to Legal Ohms by means of the known relation between the two, are denoted by an asterisk.

TABLE I.

Tube No. 37. Value given by Benoit, 1·00045.

Date and Method of Treatment.	Value in L. Ohms.
April 13, 10 A.M.	·99991*
„ 10.30	·99987
„ 11.55	·99998
„ 1 P.M. {After passing the mercury through.}	·99997
April 14	·99998
April 14 {After passing the mercury through.}	·99996
April 15	·99979
April 15	·99976*
Mean value	·99990

TABLE II.

Tube No. 38. Value given by Benoit, 1·00066.

Date and Method of Treatment.	Value in Legal Ohms.
April 8, 11.45 A.M.	1·00006
April 8, 1.30 A.M.	1·00018
April 9	1·00023*
April 9	1·00019
April 11 {Tube taken to pieces, re-cleaned and refilled.}	
April 13	1·00011*
April 13	1·00010
April 14	1·00015
April 15	1·00002
April 15	1·00003*
Mean value	1·00012

TABLE III.

Tube No. 39. Value given by Benoit, ·99954.

Date and Method of Treatment.	Value in Legal Ohms.
April 14	·99905
April 15, 10.30 A.M.	·99917
April 15, 11 A.M. {After passing the mercury through.}	·99921
April 15, 11.55 A.M.	29921
April 15, 12.15 {After passing the mercury through.}	·99921
April 15	·99917
Mean value	·99917

Thus we may collect the results as below.

TABLE IV.

No. of Tube.	Value found by Benoit.	Value found by R. T. G.	Difference.
37.....	1·00045	·99990	·00055
38.....	1·00066	1·00011	·00055
39.....	·99954	·99917	·00037
Mean.....	1·00022	·99972	·00050

Thus there is a difference of ·0005 legal ohm between the two. My results are based upon the value of the resistance of mercury in terms of the B.A. unit adopted by the British-Association Committee. If we denote by M the resistance at 0° C. of a column of mercury 1 metre long, 1 square millim. in section, then, according to the number adopted by the Committee,

$$M. = ·9540 \text{ B.A.U.}$$
$$\text{B.A.U.} = 1·04820 \text{ M.}$$

The most recent values actually found for these relations are given below:—

TABLE V.

M in B.A.U.	B.A.U. in M.	Observer.
·95412	1·04808	Rayleigh * and Sidgwick.
·95374	1·04851	Mascart†, Nerville, and Benoit.
·95334	1·04894	Strecker ‡.
·95411	1·04809	L. Lorentz §.

M. Benoit has not compared his tubes directly with the B. A. units. The mean of my comparisons gives for his mercury tubes,

$$M. = ·95348 \text{ B.A.U.}$$
$$\text{B.A.U.} = 1·04878 \text{ M.}$$

* "On the Specific Resistance of Mercury," Phil. Trans. pt. 1, 1883.

† *Journal de Physique*, June 1884.

‡ *Abh. der bayer. Akad.* Bd. xv.; also Wied. *Ann.* vol. xxv.

§ Wied. *Ann.* vol. xxv.

Thus there is a difference of ·0005 ohm between standards issued by M. Benoit as representing the legal ohm, and those issued by myself on behalf of the B. A. Committee, M. Benoit's standards being less than mine by that amount.

Some other points remain to be noticed. The first time that I filled Nos. 37 and 39, I obtained results for the resistances differing appreciably from those given above. They were filled each time with equal care, the mercury had been treated in the same manner, and there seemed no reason, taking each observation separately, why one should be better than the other. For No. 37 I found on April 8th the value 1·00044 legal ohms ; I noticed on April 9 that this had gone up to 1·00071 ; and on passing the mercury through as already described, it rose further to 1·00080. These differences are considerably greater than any possible errors of observation ; and it was clear that there was some small change progressing in the tube. It was carefully examined, but no trace of an air-bubble could be seen ; and then it was emptied, cleaned again, and refilled, with the results given in Table I. Much the same was observed with coil No. 39. No. 38 was cleaned and filled twice, but gave perfectly consistent results all through. This uncertainty seems to me to constitute one serious objection to the general employment of mercury tubes as standards.

I believe that for the observations recorded in the above Tables the tubes were properly filled, and that there were no bubbles of air in them. This, I think, is shown by the agreement between the results of different fillings, and I suppose that the resistance will not seriously alter so long as the same mercury remains in the tubes ; but it appears that, after filling, electrical experiments are required to make certain that every bubble of air has been removed, and that the tube really has its true resistance. I should imagine, too, that the tubes would require somewhat frequent refilling to make sure that the mercury may remain pure. My experiments showed that it was almost impossible to keep the inside of the cups above the mercury perfectly dry. It was necessary, when making the comparisons, to remove the glass stopper and insert the contact-pieces; and this had to be done when the tube was immersed in the ice-water.

The cups were open to the air for a short time in making this change, and that short time was quite sufficient to cause a deposit of dew to be formed on the inside of the cup. So long as this slight moisture remained in the cup and did not reach the tube itself, of course it did not affect the results; but it would be difficult to feel certain that after a time the mercury in the tube was quite dry.

This difficulty would be avoided by working at the temperature of the room rather than at that of melting ice. The large temperature-coefficient of mercury, from three to four times that of platinum-silver alloy, is, however, an objection to this.

Another difficulty, caused by the necessity of working at zero, was that the mercury in the cups was always slightly warmer, 0°·2 or 0°·3 C., than the ice. This was, no doubt, caused by the conduction of heat down the copper connecting-rods, and from the upper portions of the glass of the cups which were exposed to the air. This, of course, would necessitate a small correction to the values of the resistances given in the Tables; but the correction must be exceedingly small, for the temperature of the mercury in the tube itself, where it is actually in contact with the ice, must be zero, and it is only the small portions of the tubes which lie within the cups which will be at the higher temperature.

The necessity of using the somewhat complicated connecting pieces is also a great drawback to the practical usefulness of the mercury standards. I hope shortly to carry out some experiments on the permanence of the contact between mercury and amalgamated platinum. Contact-pieces of platinum with their end amalgamated with mercury would be easier to work with, and should give consistent results.

I have also made some observations on the variation of the resistance of mercury with temperature.

The formula quoted above from Benoit is given by Mascart, Nerville, and Benoit as determined from experiments between 0° and 100°.

It is

$$R_t = R_0(1 + \cdot 0008649t + \cdot 00000112t^2).$$

According to Strecker,

$$R_t = R_0(1 + \cdot 000900t + \cdot 00000045t^2).$$

This is derived from observations at 0°, 10°, 15°, and 20°.

Lorenz finds that between 0° and 27°·32 his experimental results agree with the formula

$$R_t = R_0(1 + \cdot 0009013t);$$

and from 8°·32 to 35°·31 with

$$R_t = R_0(1 + \cdot 000916t).$$

While Siemens and Halske give

$$R_t = R_0(1 + \cdot 0008523t + \cdot 000001356t^2).$$

I made observations on the two tubes, Nos. 37 and 39, determining their resistances at temperatures of about 0°, 5°, 10°, and 15°; and the results of the observations show that the average change for low temperatures is less than that given by the above formulæ. Thus, if we call $(R_t - R_0)/R_0 t$ the average change between temperatures t and 0, the values, as found from my experiments and as calculated by the formulæ of Siemens and Benoit, are as given below.

Average change between	Siemens and Halske.	Benoit.	R. T. G.
5° and 0°	·000858	·000870	·000834
10 „ 0	·000865	·000876	·000861
15 „ 0	·000872	·000881	·000879

The values given in the last column are the mean of those derived from the two tubes; the greatest difference between the mean and any one observed value was ·000005. Strecker's values would be above Benoit's.

Lord Rayleigh found for the average change between 0° and 12°, ·000861.

Of course the value of this temperature-coefficient depends somewhat on the glass of the tube; but the differences between the coefficients of expansion of various kinds of glass are too small to account for the whole of the differences shown above.

The paper by Herr Strecker on the same subject has been already referred to, and it will be noticed that the number he arrives at as expressing M. in B.A. units differs by ·00078 from that given by Lord Rayleigh; or, if we take the numbers expressing the B.A.U. in terms of M., the difference is ·00086. Now Strecker had compared his mercury tubes with a B.A.U. tested carefully by myself against the standards, and found to have the value of

·99937 B.A.U. at 17·7.

This coil is marked 𝔗, No. 54.

On the conclusion of his experiments, Herr Strecker returned this coil to me, and kindly sent with it a German-silver copy of his tubes marked No. 13, and said to have, according to his determinations, a resistance of

1·00189 M. at 10° C.,

with a temperature-coefficient of

·000247 per degree.

It was thus in my power to repeat my comparison of 𝔗, No. 54, with the standards, and to compare the values of M. as found by Herr Strecker with the B.A.U. The value found for 𝔗 54 agreed very closely with that given by Herr Strecker; the results of my comparison of Herr Strecker's coil with the B.A.U. are given below. The comparisons were made as usual, by Carey-Foster's method; but since the difference between the two resistances was such as to require the use of a long piece of the bridge-wire, a resistance of 20 B.A. units was introduced in multiple arc with the standard 1 unit. This resistance was composed of the two standard ten-unit coils of the Association.

Table VI.

Date.	Standard of Comparison.	B.A.U. in M.	M. in B.A.U.
April 9	Flat.	1·04814	·95407
„ 15	℔ 54.	1·04806	·95413
„ „	Flat.	1·04791	·95427
„ 16	Flat.	1·04820	·95402
„ „	℔ 54.	1·04811	·95410
„ 17	℔ 54 and 20.	1·04801	·95419
„ „	Flat and 20.	1·04824	·95399
„ „	Flat and 20.	1·04797	·95422
Mean		1·04808	·95412

These numbers, it will be seen, agree with those found by Lord Rayleigh and Lorenz, and differ greatly from Strecker's values.

They represent the results of the comparison of a German-silver copy of Strecker's tubes with the B.A. units, including the actual coil employed by Strecker, ℔ No. 54; while Strecker's numbers give the result of the comparison of his tubes directly with No. 54.

I have not been able to explain the discrepancy. Herr Strecker suggests in a letter to me on the subject, that the resistance of his coil No. 13 has changed somewhat in the year which has elapsed since his comparisons were made. At present I am in correspondence with Prof. F. Kohlrausch, of Würzburg, in whose laboratory Strecker's experiments were conducted, and hope to obtain another copy of Strecker's tubes, or at any rate to have another comparison made between his tubes and the B.A. standards.

XXVII. *Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell.* By J. A. Fleming, *M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London**.

[Plates III. & IV.]

If any number of points in a plane be joined together by linear conductors such as metallic wires, we have an arrangement of conductors which is called a Network. If at any point in the network a current of electricity be allowed to flow in and is drained off at some other point by conductors, called respectively the anode and kathode conductor, then, after a short period, depending on the self and mutual induction coefficients of the various conductors, the total quantity of electricity arriving by the anode will distribute itself throughout the network and settle down into a steady flow. When this is the case there is a certain definite difference of potential between the anode or source-point and the kathode or sink-point, and there is also a certain definite and constant strength of current in the anode conductor and in every mesh or branch of the network. Call α and γ the potentials of these source- and sink-points, and x the strength of the current in the anode lead, that is the whole quantity of electricity flowing per second through the network, then $(\gamma-\alpha)/x$ measures the resistance of the network. We can imagine the network replaced by a single linear conductor or wire of such sort that if the anode and kathode conductors are applied to its ends, the difference of potentials at the ends of this simple conductor and the strength of the current flowing through it have the same numerical values γ, α, and x. The resistance of this single conductor is then the same as that of the complex network.

The resistance of the network is obviously some function of the resistances of the separate conductors or wires which compose it, and is capable of being calculated from them. Experimentally, the resistance of a complicated network would

* Read June 27, 1885.

best be determined by the measurement of the current-strength in the anode lead and the difference of potential between the source and the sink. Theoretically, it is interesting to examine the law of distribution of currents in a network, and to reduce to a function of the separate resistances the total resistance of the whole network between any two points.

§ 2. In his larger Treatise on Electricity, Clerk Maxwell has treated the general case to determine the differences of potentials and the currents in a linear system of n points connected together in pairs by $\frac{1}{2}n(n-1)$ linear conductors*, and has shown how to form the linear equations, the solution of which gives the condition of the network when given electromotive forces acting along some or all of the branches have established steady currents in them.

The usual method of obtaining a solution for the distribution of currents is the application of Ohm's law round the several currents of the network, controlled by the condition of continuity that there is no creation nor destruction of electricity at the junctions.

Since the publication of the first edition of his Treatise, Maxwell reduced these two sets of equations to one set by the simple device of regarding the real currents in the meshes of the network as the *differences* of imaginary currents round each cycle or mesh of the network, all directed in the same direction, and thus obtained by the application of Ohm's law a single set of linear equations, the solution of which gives the required currents in each branch. Maxwell's method is as follows†:—If we have p points in space and join them together by lines, the least number of lines which will connect all the points together is $p-1$. If we add one line more we make a closed circuit somewhere in the system; that is

* 'A Treatise on Electricity and Magnetism,' 2nd edition, Vol. i. § 280 and § 347.

† This method was first given by Clerk Maxwell in his last course of University lectures. It is alluded to in the second edition of his larger Treatise and in the Appendix of his smaller Treatise by their respective editors, Mr. W. D. Niven and Professor Garnett, to whom it was communicated by the present writer.

to say, a portion of space is enclosed and forms a cell cycle or mesh. Every fresh line added then makes a fresh mesh; and hence if there are l lines altogether joining p points, the number of cycles or cells will be $k=l-(p-1)$. Now let such a system of points and lines represent conducting wires joining fixed points, and forming a conducting network. Let a symbol be affixed to each point which represents the electrical potential at that point, and also a symbol affixed to each line representing the electrical resistance of the conductor represented by it. In such a diagram of conductors the form is a matter of indifference so long as the connections are not disturbed and lines are not made to cross unless the conductors they represent are in contact at that point.

Consider a network, Pl. III. fig. 1, formed by joining nine points by thirteen conductors. Then there will be $13-(9-1)=5$ cycles or cells. Now let an electromotive force E act in one branch B, and give rise to a distribution of currents in the network. Let α, β, γ, δ, &c. represent the potentials at the points, and A, B, C, D, &c. the electrical resistances of the conductors joining these points, and imagine that round each cycle or circuit an imaginary current flows, all such currents flowing in the same direction.

A circuit is considered to be circumnavigated positively when you walk or go round it so as to keep the boundary on your right hand. Hence, going round an area A in the direction of the arrow is positive as regards the inside if you walk inside the boundary-line, and negative as regards external space B if you walk in the same direction round the outside. We shall consider a current, then, as positive when it flows round a cycle in the opposite direction to the hands of a watch. Returning then to the network, we consider that round each cycle flows an imaginary current in the positive direction. The real currents in the conductors are the differences of these in adjacent cycles or meshes, and the imaginary currents will necessarily fulfil the condition of continuity, because any point is merely a place *through* which imaginary currents flow, and at which therefore there can be no accumulation nor disappearance of electricity.

Let x, y, z, &c. denote these imaginary like-directed currents. Then $x-y$ denotes the real current in the branch I,

and similarly $x-z$ that in branch H. Then x, y, z, &c. may be called the cyclic symbols of these areas. The cyclic symbol of external space is taken as zero; hence the real current in branch B is simply x.

Let an electromotor act on the branch B, bringing into existence an electromotive force in that branch. Let the internal resistance of the electromotor be included in the quantity B, representing the resistance of the branch A. Then apply Ohm's law to the cycle x formed by the conductors B, I, H; we have

$$E-Bx=\gamma-\alpha.$$

x is the actual current in this case flowing in the resistance B, and the potential at the ends of B is equal to the effective electromotive force acting in it *less* the product of the resistance of the conductor multiplied by the current flowing in it. For the conductor I we have similarly

$$\gamma-\beta=(x-y)I.$$

Hence $x-y$ represents the actual current in I: it is the difference of the imaginary currents flowing round the x and y cycles in the positive direction. And for the conductor H we have also

$$\beta-\alpha=(x-z)H.$$

Add together these three equations,

$$E=\gamma-\alpha+Bx,$$
$$O=\beta-\gamma+(x-y)I,$$
$$O=\alpha-\beta+(x-z)H\ ;$$

and we have, as the result of going round the cycle x formed of conductors B, I, and H,

$$E=x(B+I+H)-yI-zH. \quad . \quad . \quad . \quad . \quad (1)$$

α, β, γ have disappeared in virtue of these opposite signs.

This equation (1) is called the equation of the x cycle; and we see that it is formed by writing as coefficient of the cyclic symbol x the sum of all the resistances which bound that cycle, and subtracting the cyclic symbol of each neighbouring cycle multiplied respectively by the common bounding resistance as coefficient, and equating this result to the effective electromotive force acting in the cycle, written as positive or nega-

tive according as it acts with or against the imaginary current in the cycle. This is Maxwell's rule.

Since there are k cycles or meshes we can in this way form k independent equations, and by the solution of these determine the k independent variables, x, y, z, &c. The value of the current in any branch is then obtained by simply taking the difference of these variables belonging to the adjacent meshes, of which the conductor or branch considered is the common boundary.

§ 3. Let us now consider the most general case possible, in which we have a network composed of linear conductors sufficiently far apart to have no sensible mutual induction, and let there be electromotive forces acting in each branch or conductor. Let the system be considered to have arrived at the steady condition. Let x, y, z, &c. be the cyclic symbols or measure of the imaginary current circulating counterclockwise round each mesh. Let A, B, C, &c. (fig. 3) be the resistances, and e_1, e_2, e_3, &c. the electromotive forces acting in each branch. These are reckoned positive when they tend to force a current round the mesh counterclockwise, and negative when they act in the opposite direction. Then the equation to the x cycle will be

$$x(\mathrm{A}+\mathrm{J}+\mathrm{L})-y\mathrm{J}+0z+0u+0w=e_1.$$

The symbols of all the cycles are written down, putting in those of z, u, and w with zero coefficients, as they are not adjacent cycles to that of x. We shall have five equations similar to the above for the other cycles, y, z, w, and u.

Now it can very simply be shown from the theory of determinants, that if there are n linear equations of the type

$$\begin{array}{l}
a_1x_1+a_2x_2+\ .\ .\ .\ .\ .\ .\ .\ a_nx_n=p_1,\\
b_1x_1+b_2x_2+\ .\ .\ .\ .\ .\ .\ .\ b_nx_n=p_2,\\
\ \ .\ \ \ \ .\ \ \ \ \ \ \ \ \ \ .\\
\ \ .\ \ \ \ .\ \ \ \ \ \ \ \ \ \ .\\
\ \ .\ \ \ \ .\ \ \ \ \ \ \ \ \ \ .\\
\ \ .\ \ \ \ .\ \ \ \ \ \ \ \ \ \ .\\
\ \ .\ \ \ \ .\ \ \ \ \ \ \ \ \ \ .\\
k_1x_1+k_2x_2+\ .\ .\ .\ .\ .\ .\ .\ k_nx_n=p_n,
\end{array}$$

the solution for any variable x_1 is the quotient of the determinants

$$x_1 = \frac{\begin{vmatrix} p_1 & a_2 & \cdots & a_n \\ p_2 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ p_n & k_2 & & k_n \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ k_1 & k_2 & & k_n \end{vmatrix}}$$

The only difference between the numerator and denominator is that the solution for x_n is given by writing as numerator the determinant of the n equations having the column $p_1, p_2 \cdots p_n$ substituted for its nth column, and then writing down as denominator the determinant of the n equations simply.

Thus, for example, the solution of the three linear equations

$$ax + by + cz = d,$$
$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$

is

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

with similar expressions for y and z, differing only in having as numerators respectively

$$\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix},$$

denominator being the same.

In this case the evaluation of these determinants is easy: a

simple symmetrical process of taking products, according to the rule,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ecg + bdi + afh).$$

§ 4. The properties of determinants enable us, however, very easily to evaluate a numerical determinant of any order. The process consists in the gradual reduction of the determinant in order by such transformations as will render all the elements of the first row or column zero except the first. The determinant is then reduced to the product of its leading elements and the corresponding minor. A repetition of this lowers the determinant one degree at each stage ; and finally, when it is resolved into a numerical two-row determinant, a simple cross multiplication gives its value.

The process of evaluation of a numerical determinant is dependent on four principles :—

(1) That the value of a determinant is not altered if rows are changed into columns.

(2) The interchange of two rows or two columns reverses the sign of the determinant.

(3) If every constituent in any row or column be multiplied by the same factor, then the determinant is multiplied by that factor.

(4) A determinant is not altered if we add to each constituent of any row or column the corresponding constituents of any other row or column multiplied respectively by an identical factor, positive or negative.

For example, suppose that the solution of a series of network equations with numerical coefficients of resistance yield the determinant

$$\begin{vmatrix} 5 & 3 & 1 & 6 \\ 7 & 8 & 9 & 2 \\ 2 & 1 & 4 & 3 \\ 10 & 7 & 5 & 7 \end{vmatrix}$$

we proceed to operate on this as follows :—Subtract the second column from the first and write the remainder. As a new first column we get

$$\begin{vmatrix} 2 & 3 & 1 & 6 \\ -1 & 8 & 9 & 2 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Subtract the third row from the first and put the remainder as a new first row. Also add the third row to the second for a new second row, and we get

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Again, subtract the first row from the third for a new third, and subtract three times the first row from the fourth row for a new fourth row, and we have

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 0 & -1 & 7 & 0 \\ 0 & 1 & 14 & -2 \end{vmatrix}$$

which is equivalent to the third order determinant

$$\begin{vmatrix} 9 & 13 & 5 \\ -1 & 7 & 0 \\ 1 & 14 & -2 \end{vmatrix}.$$

And a similar series of operations reduces this to

$$\begin{vmatrix} 76 & 5 \\ 21 & -2 \end{vmatrix}$$

which is equal to

$$-76\times 2-5\times 21=-257.$$

Accordingly a series of simple subtractions and multiplications will effect the evaluation of any numerical determinant, and enable us to solve a series of linear network equations for the currents in all the branches when the numerical values of the resistances of the conductors are given. The equations as written above give as solutions the values of the cyclic symbols or imaginary currents round each mesh. To obtain the actual current in any branch, we should have to obtain the values of the cyclic symbols or imaginary currents, for the adjacent meshes of which the given branch is a common

boundary. Maxwell ingeniously saves labour in this operation by taking as the symbol for one mesh say $x+y$, and for an adjacent mesh y (fig. 4), and then the real current in the branch AB is

$$x+y-y=x.$$

And the simple rearrangement and solution of the network equation gives at once as value for x the current in the resistance AB, which is the common partition of the two meshes.

§ 5. Returning now to the case when there is only one impressed electromotive force in one branch, we see that in forming the cycle equations only one will be equated to an electromotive force, viz. the equation for the mesh containing the impressed electromotive force in one of its branches. All the other equations will be equated to zero ; and accordingly the equation for the current in any conductor will be of the form

$$x=\frac{\mathrm{E}\Delta_{n-1}}{\Delta_n};$$

where Δ_n is a determinant of the nth order, and Δ_{n-1} is a first minor of this. Referring to fig. 1, we see that, by writing down the five equations of the cycles x, y, z, u, w, we obtain equations by which to calculate the currents in any of the thirteen branches, and the current in branch B will be

$$x=\frac{\mathrm{E}\Delta_{n-1}}{\Delta_n};$$

where Δ_n is the determinant formed of the coefficients of the five equations, and Δ_{n-1} is the first minor corresponding to the coefficient of x in the equation of the x-cycle.

We also saw that if γ and α are the potentials at the ends of the branch B, $\mathrm{E}-\mathrm{B}x=\gamma-\alpha$.

Now consider that part of the network which remains if the conductor B is removed, and let us imagine that a current n continues to be forced into it at γ and drained out at α ; the total resistance of that part of the network, not counting B, is

$$\frac{\gamma-\alpha}{x};$$

but this is equal to

$$\frac{\mathrm{E}}{x}-\mathrm{B}.$$

Now since the resistance of B may be anything, let it be zero;

then the total resistance of the network between γ and α will be

$$R = \frac{E}{x};$$

but

$$x = \left[\frac{E\Delta_{n-1}}{\Delta_n}\right]_{B=0},$$

where the suffix and bracket denote that after the determinants are formed from the cycle equations, according to Maxwell's rule, then in them B is put equal to zero.

If we denote the determinant of all the n-cycle equations under the condition of $B=0$ by d_n, and by d_{n-1} the first minor of this or the minor of its leading element corresponding to the coefficient of x with the resistance of the circuit containing the effective electromotive force put equal to zero, we have for the total resistance R of the network between the points at which the current enters and leaves, the expression

$$R = \frac{d_n}{d_{n-1}}.$$

Since, then, as we have seen, the linear equations for the cycles can always be solved by evaluating the determinants, it follows that in all cases, no matter how complicated, the resistance of any network can be calculated by simple arithmetic processes from the given resistances of the branches or conductors which compose it. We have therefore an interesting extension of Maxwell's method of calculating the currents in a network and the potentials at the junctions to a method of calculating the combined resistance of a number of conductors forming a network; which method consists, as seen above, in forming a certain determinant whose elements are formed of the separate resistances of the branches, and dividing this determinant by another of an order next below, viz. the first minor of its leading elements; and we find that the resistance between any two points of any network of conductors, however complicated, is expressible as the quotient of a certain determinant by another formed from it.

§ 6. We shall proceed to illustrate this method by a few examples.

1. Find the resistance between the points 1 and 3 (fig. 5) of a network consisting of five conductors, whose resistances are A, B, C, D, E, joining four points, 1, 2, 3, and 4.

Connect 1 and 3 by an imaginary conductor of zero resistance, and having an electromotive force, e, supposed to act in it. Let x, y, z denote the cycles or imaginary like-directed currents in the three meshes so formed, and write down the current equations, according to Maxwell, for these three cycles:—

$$\begin{aligned} (A+B)x \quad & -Ay \quad & -Bz \quad & = e, \\ -Ax \quad & +(A+E+D)y \quad & -Ez \quad & = 0, \\ -Bx \quad & -Ey \quad & +(B+C+E)z & = 0. \end{aligned}$$

Then, by what has been shown above, the resistance R between the points 1 and 3 of the network is given by the expression

$$R = \frac{\begin{vmatrix} (A+B), & -A, & -B \\ -A, & (A+E+D), & -E \\ -B, & -E, & (B+C+E) \end{vmatrix}}{\begin{vmatrix} (A+E+D,) & -E \\ -E, & (B+C+E) \end{vmatrix}}$$

In dealing with numerical cases we need no longer introduce any notice of imaginary electromotive forces, but proceed according to the following rule.

To determine the resistance of a network of conductors between any two points on the network. Join these two points by a line whose resistance is supposed zero, and give symbols to the meshes of the network so formed; calling this additional mesh produced by the added zero conductor the *added mesh*. Then write down a determinant whose dexter diagonal has for elements the sum of the resistances which bound each mesh, beginning with the added mesh; and for the other elements of each row the resistances which separate this mesh respectively from adjacent meshes, and having the minus sign prefixed, zeros being placed for elements corresponding to nonadjacent meshes.

More explicitly, if we denote by x, y, z, &c. the meshes, x being the added mesh, and by ΣR_x, ΣR_y, ΣR_z, &c. the sum of the resistances which bound each cycle, then these will be the elements along the dexter diagonal of the determinant.

And if x and y are adjacent meshes, and ${}^{x}\mathrm{R}$ represents the resistance of the common boundary, then $-{}^{x}\mathrm{R}$ will be the element in the xth row and yth column, and also in the yth row and xth column; but if x and z are nonadjacent meshes, then 0 will be the element in the xth row and zth column, and also in the zth row and xth column. Having formed this determinant, which we call the network determinant, we divide it by the first minor of its leading element; and the quotient is the resistance of the network between the two points, joined by the zero-conductor forming the added mesh. It is seen that, owing to the mode of formation of the network equations, the network determinant is a symmetrical determinant—that is, one half of the determinant is the reflection, as it were, of the other half in the diagonal considered as a mirror.

§ 7. As a means of comparing the results of this method with other known results, let us take the exceedingly simple case of three conductors joining two points in what is commonly called multiple arc.

Let 1, 2, and 3 (fig. 6) be the three conductors joining two points A and B; let their respective resistances be r_1, r_2, r_3; then join A, B by a dotted line so as to make one added mesh, and let the resistance of this added circuit be zero. Then, without writing down the equations to the cycles, we see that the network determinant is

$$d_n = \begin{vmatrix} r_1 & -r_1 & 0 \\ -r_1 & r_1+r_2 & -r_2 \\ 0 & -r_2 & r_2+r_3 \end{vmatrix}$$

The elements r_1, r_1+r_2, r_2+r_3 of the dexter diagonal are the sums of the resistances which bound each mesh, x, y, and z, taking the added mesh x first.

The other elements of the first row are the resistances, with minus sign prefixed, which separate the mesh x from mesh y and mesh z; or are common to x and y and x and z, viz. r_1 and zero, because x and z are nonadjacent. And, similarly, if m and n are any two meshes, then the element in the nth row and mth column is the resistance separating or common to the two meshes; and the element in the nth row and mth column is identical with that in the mth row and nth column:

zero being placed as an element if these meshes, m and n, have no common boundary or circuit.

The above determinant is easily evaluated. By adding the first row to the second for a new second row, and this new second row to the third for a new third row, we transform the determinant easily into

$$\begin{vmatrix} r_1 & -r & 0 \\ 0 & r_2 & -r_2 \\ 0 & 0 & r_3 \end{vmatrix}$$

which is equal to

$$r_1 r_2 r_3.$$

The first minor of the leading term of the network determinant is

$$\begin{vmatrix} r_1+r_2 & -r_2 \\ -r_2 & r_2+r_3 \end{vmatrix} = d_{n-1},$$

which is equal to

$$r_1 r_2 + r_2 r_3 + r_3 r_1 ;$$

and hence the resistance of the network between A and B is

$$\frac{d_n}{d_{n-1}} = \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1},$$

which is a known result. In these simple cases the above general rule is, of course, a less easy method of finding the combined resistance than the direct application of Kirchhoff's corollaries of Ohm's law ; but whereas the general method is alike applicable to the most complicated as well as to the most simple cases, the simple direct method requires twice as many equations, and does not determine the direction as well as magnitude of the current in each branch.

§ 8. As a simple numerical example we may take the case of a crossed square of wires. Let 12 conductors join 9 points (fig. 7) so as to form a square divided into four squares, or a four-mesh network of conductors. Let the resistance of each branch, as *ab*, be unity. It is required to find the combined resistance between A and B. Number the meshes 1, 2, 3, 4, 5 ; 1 being the added mesh formed by joining A B by a dotted line, making an additional fifth mesh, the resistance of this additional ideal conductor being zero. Then the network

determinant is

$$\begin{vmatrix} 4 & -1 & -2 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -2 & -1 & 4 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 4 \end{vmatrix} = d_n.$$

The dexter diagonal has for each element 4, viz. the sum of the four resistances, each to unity, which form each mesh or cell. And all the other figures, say, in the nth row, are the resistances (with minus sign prefixed) separating the nth mesh from all other meshes, zero being placed in the column corresponding to any mesh which has no common conductor or branch with this nth mesh. The order in which the columns stand and also the rows correspond to the order in which the meshes are numbered in fig. 7.

The numerical value of this determinant is easily found to be $288 = 3 \times 96 = d_n$. Now if we take the first minor of its leading element, we get a determinant formed of the elements included in the dotted rectangle ; and taking this as a separate determinant and evaluating it, we have its value

$$d_{n-1} = 192 = 2 \times 96 ;$$

hence the resistance of the network between the points A and B is

$$\frac{d_n}{d_{n-1}} = \frac{288}{192} = 1\tfrac{1}{2} \text{ units.}$$

§ 9. One more simple numerical case may be taken and compared with the results of known methods.

Let a hexagon of conductors be taken (fig. 8) having crossed diagonals all meeting in the centre. Let the resistance of each side, as ab, be unity, and also let the resistance of each semidiagonal, as $0a$, be unity. Then required the combined resistance of this network of 12 conductors between the points A and B diametrically opposite. Join the points A and B by a dotted line of zero resistance, making an added mesh 1.

Mark the other meshes 2, 3, 4, 5, 6, 7. Then by forming the network equations it is easily seen that the network determinant d_n is

$$\begin{vmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix} = d_n.$$

The value of this determinant is 256.

The first minor of the leading element of d_n is d_{n-1}.

$$= \begin{vmatrix} 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix}$$

The value of this last is 320.

Hence the resistance of the network between the points A and B is

$$R = \frac{d_n}{d_{n-1}} = \frac{256}{320} = \frac{4}{5}.$$

We can easily verify this result in the above symmetrical case, for the hexagonal framework in fig. 8 is traversed symmetrically by the current flowing through it; and hence no disturbance of the distribution of currents will take place by separating it, as in fig. 9. We break the connection between the semidiagonal conductors a, b and the mean diagonal A B, whilst keeping them in contact with each other, the resistance of each branch still remaining unity. It is then easily seen that the hexagon so arranged must offer exactly the same resistance between the points A and B as in its original form.

Now the combined resistance of a, b, and f, each equal to unity, between the points C, D is $\frac{2}{3}$, and the combined resistance of this with e and g in series is $2\frac{2}{3}$; and hence the total resistance of the whole network between A and B is equal to that of three conductors in multiple arc whose resistances are

respectively $2\frac{2}{3}$, 2, and $2\frac{2}{3}$, which is equal to

$$\frac{1}{\frac{1}{2\frac{2}{3}}+\frac{1}{2}+\frac{1}{2\frac{2}{3}}}=\frac{4}{5},$$

the same result as obtained above.

These numerical examples show conclusively that, in cases in which the resistance of a network can be obtained by simple direct methods, the results coincide, as should be the case, with those obtained by the employment of the general method; but at the same time the general method is capable of conducting easily to a solution in the most unsymmetrical cases. The general rule will, for instance, just as easily give the determinants when the selected points between which the resistance is required are not symmetrically placed, but are, say, adjacent angles of the hexagon, in which case no such simple direct method as employed above can be used.

§ 10. The following example will give a good illustration of Maxwell's method of treating network problems, viz. the case of Sir W. Thomson's resistance-balance for small resistances. In this arrangement (fig. 10) 9 conductors join 6 points and form 4 cells. B is the battery-circuit in which operates an electromotive force E. Let the four cycle currents be denoted by $x+y$, y, z, and w. These are the imaginary like-directed currents round the circuits, and the real currents in the branches are the differences of these.

The problem is to determine the current in the galvanometer branch G, and the relation of the resistances when this current through G is zero. Let P, Q, S, T, R, r, D be respectively the resistances of the branches, and G the resistance of the galvanometer circuit, and B the resistance of the battery circuit. Then $x+y$ and y being the imaginary like-directed currents in the two adjacent meshes of which the galvanometer branch is the common boundary, then $x+y-y=x$ is the current through the galvanometer.

Proceeding to write down the cycle equations, according to Maxwell's rule, we have

$$\begin{aligned}
(\mathrm{P}+\mathrm{G}+\mathrm{Q}+\mathrm{R})\overline{x+y}-\mathrm{G}y-\mathrm{Q}z-\mathrm{R}w&=0,\\
(\mathrm{T}+r+\mathrm{S}+\mathrm{G})y-\mathrm{G}\overline{x+y}-\mathrm{S}z-rw&=0,\\
(\mathrm{Q}+\mathrm{S}+\mathrm{D})z-\mathrm{S}y-\mathrm{Q}\overline{x+y}-\mathrm{D}w&=0,\\
(\mathrm{R}+\mathrm{D}+r+\mathrm{B})w-\mathrm{R}\overline{x+y}-\mathrm{D}z-ry&=\mathrm{E}.
\end{aligned}$$

Rearranging these equations and solving for x, we have the following value :—

$$x = \frac{E}{\Delta} \begin{vmatrix} -\overline{Q+S}, & D, & -D \\ T+S+r, & T+r, & -r \\ P+Q+R, & P+R, & -R \end{vmatrix},$$

in which Δ is the determinant of the four equations in x, y, z, and w, and whose specific value does not concern us.

This gives the current in the galvanometer-branch ; and if this is zero, then the determinant in the numerator of the equation giving x must be zero. Hence, when x is zero, we have

$$\begin{vmatrix} -\overline{Q+S}, & D, & O \\ T+S+r, & r+T, & T \\ P+Q+R, & P+R, & P \end{vmatrix} = 0,$$

this determinant being derived from the one in the equation for x by adding the second and third columns for a new third column.

This last determinant equation writes out into

$$(Q+S+D) \begin{vmatrix} T, & r \\ P, & R \end{vmatrix} + D \begin{vmatrix} T, & S \\ P, & Q \end{vmatrix} = 0.$$

Hence the condition that the current in the galvanometer-branch shall be zero is that both determinants in this expression shall be simultaneously zero, or

$$\begin{vmatrix} T, & r \\ P, & R \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} T, & S \\ P, & Q \end{vmatrix} = 0 ;$$

that is,

$$\frac{T}{P} = \frac{r}{R} = \frac{S}{Q}.$$

Hence this condition expresses the relation which must hold good between the magnitudes of the resistances T, P, Q, S, r, R, in order that the galvanometer-branch G may be conjugate to the battery-branch B.

The above example shows well the symmetry of the method when dealing with a case of distribution of currents in a network.

§ 11. As a final illustration, let us consider the case of a circular wire A P B Q, with a diametral wire P Q across it.

Take any two points A, B, at the extremities of a diameter

not coinciding with P Q, but separated by an angular distance θ from it, and let us obtain the resistance of the circular wire so crossed between the points A and B.

Join the points A, B by a dotted line of zero resistance. Call the three meshes so formed x, y, and z; let r be the radius of the circle; and let ρ be the electrical resistance of the wire per unit of length. Then the

$$\begin{aligned} &\text{Resistance of branch PQ} = 2\rho r, \\ &\quad\text{,,}\qquad\text{,,}\qquad \text{AP} = \rho r\theta, \\ &\quad\text{,,}\qquad\text{,,}\qquad \text{AQ} = \rho r(\pi-\theta), \end{aligned}$$

and

$$\begin{aligned} &\text{Resistance of branch BQ} = \text{resistance of AP}, \\ &\quad\text{,,}\qquad\text{,,}\qquad \text{PB} = \quad\text{,,}\quad\text{,,}\quad \text{AQ}. \end{aligned}$$

Then the network determinant d_n is

$$\begin{vmatrix} \rho r\pi, & -\rho r(\pi-\theta), & -\theta \\ -\rho r(\pi-\theta), & \rho r(\pi+2), & -\rho r2 \\ -\theta, & -\rho r2, & \rho r(\pi+2) \end{vmatrix}.$$

Removing the common factor $r\rho$, we have to evaluate

$$\begin{vmatrix} \pi & -\overline{\pi-\theta} & -\theta \\ -\overline{\pi-\theta} & \overline{\pi+2} & -2 \\ -\theta & 2 & \overline{\pi+2} \end{vmatrix}$$

This is very easily reduced to

$$\pi \begin{vmatrix} 0 & \theta & \pi \\ -1 & 1 & 2 \\ -\theta & 2 & \pi \end{vmatrix},$$

which is equal to

$$2\pi(\pi+\theta\pi-\theta^2),$$

and therefore

$$d_n = r^3\rho^3 2\pi(\pi+\theta\pi-\theta^2).$$

The value of the minor of the leading element of the network determinant, viz. d_{n-1}, is

$$r^2\rho^2\{(\pi+2)^2-4\} = r^2\rho^2\pi(\pi+4).$$

Hence the resistance of the network between A and B, $=$R, is

$$\begin{aligned} \frac{d_n}{d_{n-1}} = \mathrm{R} &= r\rho\frac{2\pi(\pi+\theta\pi-\theta^2)}{\pi(\pi+4)} \\ &= r\rho\,\frac{2\pi+2\pi\theta-2\theta^2}{\pi-4}. \end{aligned}$$

We can check this in the extreme cases when $\theta=0$ or $\theta=\frac{\pi}{2}$. When $\theta=0$, the network-resistance is simply that of three conductors whose resistances are $2\rho r$, $\pi\rho r$, and $\pi\rho r$ joined in multiple arc, as in Plate IV. fig. 12, because PQ now coincides with AB; and this is simply $r\rho\frac{2\pi}{\pi+4}$. It is seen at once that the above value for R becomes this when θ is put equal to zero. Now, when $\theta=\frac{\pi}{2}$, the diameter PQ joins points at equal potential (fig. 13), and is not traversed by any current at all; and hence its removal will not affect the resistance between the points A and B.

Hence the resistance of the network simply reduces to that of a circle measured at the ends of a diameter, or to two conductors of resistance $\pi\rho r$ joined in multiple arc, and this is equal to $\rho r\frac{\pi}{2}$. By putting $\theta=\frac{\pi}{2}$ in the general solution for R above, we get it reduced to $r\rho\frac{\pi}{2}$; and accordingly this formula agrees, as it should do in these reduced cases, with the results of the direct method based on first principles. If a value of θ be found which will make the expression

$$2\pi+2\pi\theta-2\theta^2$$

equal to $\pi+4$, then for such a position of the diameter AB relatively to PQ the resistance of the circle and its diagonal PQ would be exactly equal to the resistance of half the diametral wire or to its radius, assuming both the circle and diagonal to be made of wire of equal conductivity per unit of length. To find the value of θ for which this is the case, we have to solve the quadratic

$$2\pi+2\pi\theta-2\theta^2=\pi+4.$$

If we put $\theta=\frac{\pi}{180}x^\circ$, where x° is the number of degrees equivalent to the angle θ, we find, as a solution for this quadratic, that the positive root is nearly

$$171^\circ\cdot804.$$

Now 3 radians, or 3 unit-angles in circular measure, are nearly

$$171^\circ\cdot887.$$

Hence, for a position of the diagonal PQ as in fig. 14, when the arc AP is nearly equal to $\pi-3$, or to the fractional part of π, the resistance of the circle and diagonal PQ measured between the points A, B is very nearly equal to that of half the diagonal PQ; or, which is the same thing, the resistance of PQ alone is nearly double the combined resistance of the circle and diagonal measured between the points A and B at the extremity of a diameter removed 171°·804 from PQ.

§ 12. A small practical application of this last example may be made in constructing a variable resistance.

Let PAQB (fig. 15) be a narrow circular canal cut in a slab of wood or ebonite and filled with mercury. Let PDQ be a bent copper wire balanced on a pivot CD, and having its ends P and Q dipping in the trough at opposite extremities of a diameter of the circular trough PAQB.

The total resistance between any two points A and B in the trough, which are also diametrically opposite, can be varied within limits by changing the position of PQ relatively to AB.

When PQ is turned so that it is at right angles to the diameter AB, it does not affect the total resistance between A and B, and may be removed. The resistance is then just that of the circular band of mercury taken at opposite extremities of its diameter. When PQ is coincident with AB it reduces the resistance, and in intermediate positions the joint resistance of trough and diagonal wire is intermediate between the greatest and least when it is in position removed either 90° or 0° from AB.

By using a circular glass canal filled with sulphate-of-zinc solution, and a zinc diagonal electrode and amalgamated-zinc electrodes at A and B, a variable resistance may be constructed capable of being varied over considerable ranges perfectly gradually and with no imperfect contacts.

§ 13. Having illustrated, by the foregoing examples, the methods of calculating both the currents in and resistances of networks of any complexity, we return for a moment to some general considerations.

Consider a function formed of the sum of each separate resistance in a network multiplied by the square of the current strength flowing through it. This expresses the heat generated per second in the whole network by that distribution of

current. This is called the Dissipation Function of the network. It represents the rate at which energy is being transformed into heat or rendered unavailable.

Write down the dissipation function for the network in fig. 1. Call it H. Then

$$H = Bx^2 + I\overline{x-y}^2 + H\overline{x-r}^2 + Cy^2 + L\overline{z-y}^2 + Az^2 + J\overline{u-y}^2 + K\overline{z-w}^2 + (D+E)u^2 + (F+G)w^2 + M\overline{u-w}^2.$$

Now the cycle equation for the cycle or mesh y is, by Maxwell's rule,

$$(C+I+L+J)y - Ix - Lz - Ju = 0,$$

which is the same as

$$Cy - I\overline{x-y} - L\overline{z-y} - J\overline{u-y} = 0.$$

And this is at once seen to be identically the same as the first partial differential of the dissipation function with respect to the cyclic symbol y, or is the same as

$$\tfrac{1}{2}\frac{\partial H}{\partial y} = 0,$$

where ∂ represents partial differentiation; and by writing down the other cycle equations for each cyclic symbol or imaginary current, x, y, z, &c., we can show that these current-equations are respectively

$$\tfrac{1}{2}\frac{\partial H}{\partial x}, \quad \tfrac{1}{2}\frac{\partial H}{\partial y}, \quad \tfrac{1}{2}\frac{\partial H}{\partial z}, \text{ \&c.},$$

each equated to the effective electromotive force in that cycle or mesh.

Let us assume now that x is constant, but that y, z, u, w, &c. are independent variables and are arbitrarily changed. This is equivalent to supposing that a given quantity of electricity per second is pushed into the network, but that its distribution is supposed to be varied. We see that the equations which we write down, according to Maxwell, to determine the *real* distribution of currents in the network, according to Ohm's law, are the same equations as would be written down to find the values of y, z, u, w, &c., which make the dissipation function a minimum under fixed conditions of total current flowing into the network, viz. equating to zero the first partial differentials of H with respect to the variables y, z, u, &c. The same holds good generally, hence we see that this is

another way of arriving at the theorem of which Maxwell has given a proof on page 375, § 284, vol. i. of his large Treatise, 2nd edition, viz.:—"In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current."

The exact proof that the partial differentials of the dissipation function equated to zero gives the condition that the dissipation function shall be a *minimum* is not complete without an examination of Lagrange's conditions. It is obvious that the second partial differentials of the dissipation function are quantities which are resistances, viz. the coefficients of the current symbols in the cycle equations, and that the conditions for a minimum are complied with, since $\frac{\partial^2 H}{\partial y^2}$, &c. are positive; and the *discriminant* of the quadratic function of the currents or symmetrical determinants formed of these second partial differentials is what has been called above the network determinant. This and all its successive minors are positive quantities*.

§ 14. In the foregoing sections the problems have been treated under the limitations that the various meshes of the network of conductors have no mutual and no self-induction. The introduction of these inductive actions will affect in a considerable way the treatment of the problem; and the distribution of the currents in, and the resistance of, the network will be affected by them during the time taken by the currents to become steady.

In those pages of his Treatise in which Clerk Maxwell worked out his splendid dynamical theory of electromagnetism, he starts with the explanation of the methods Lagrange and Hamilton employed to bring pure dynamics under the power of analysis, and the results of Lagrange are embodied in the equation

$$X = \frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx},$$

* See Williamson's 'Differential Calculus,' p. 408, "On the Conditions for a Maximum and Minimum of a Function of any number of Variables," § 163, and Appendix.

in which X is the impressed force tending to increase the variable x, and T denotes the visible energy of the system of bodies at that instant.

This equation establishes a relation between the kinetic energy of a material system at any instant, the force impressed upon it in a certain direction, and a quantity called a variable, which expresses the state or condition of the system with respect to that direction. Maxwell, by a process of extraordinary ingenuity, extended this reasoning from materiomotive forces, masses, velocities, and kinetic energies of gross matter to the electromotive forces, quantities, currents, and electrokinetic energies of electrical matter, and in so doing obtained a similar equation of great generality for attacking electrical problems.

In the electrical problem the variables are the quantities of electricity x, y, z, &c. which have from the beginning of the epoch flowed past any points, and the analogues of the velocities are the fluxes of these, $\dot{x}$, $\dot{y}$, $\dot{z}$, &c., or the currents.

The electrokinetic energy is measured by the quadratic expression

$$T = \tfrac{1}{2}L_1\dot{x}_1^2 + \tfrac{1}{2}L_2\dot{x}_2^2 + \ \ldots\ M_{12}\dot{x}_1\dot{x}_2 +, \text{ &c.},$$

where the coefficients L_1, L_2, M_{12} are functions of the geometrical variables, but into which the electrical variables do not enter.

If now, as before, $\dot{x}_1$, $\dot{x}_2$ represent the imaginary like-directed currents round each mesh of a network, in which currents are beginning to flow, then

$$\frac{dT}{d\dot{x}_1} \text{ and } \frac{dT}{d\dot{x}_2}, \text{ &c.}$$

represent the electrokinetic momenta of these circuits. Denote them by p_1, p_2, &c., and accordingly

$$p_1 = L_1\dot{x}_1 + M_{12}\dot{x}_2, \text{ &c.}$$

If E is the impressed electromotive force in the circuit or mesh arising from some cause, battery, thermopile, dynamo machine, &c., which would produce a current independently of magneto-induction, then, if R be the total resistance round the mesh, and $\dot{x}$ the cyclic current, $R\dot{x}$ is the electromotive force required to overcome the resistance of the circuit,

and $E - R\dot{x}$ is the electromotive force available for changing the electric momentum of the circuit.

Accordingly, by Lagrange's equation,

$$E - R\dot{x} = \frac{dp}{dt} - \frac{dT}{dx},$$

where T is the electrokinetic energy. As T does not contain x, that is to say it is a function of currents, not quantities, the last term disappears, and we have

$$E - R\dot{x} = \frac{d}{dt}\frac{dT}{d\dot{x}},$$

or

$$\frac{d}{dt}\frac{dT}{d\dot{x}} + R\dot{x} = E.$$

The electromotive force is therefore expended in two things: first, overcoming the resistance R; and, secondly, increasing the electromagnetic momentum p. Now if there is no electromagnetic momentum, we have seen that the cyclic equations are of the form

$$\tfrac{1}{2}\frac{dH}{d\dot{x}} = E',$$

where H is the dissipation function of the system, and E' is the acting electromotive force concerned in overcoming the resistance of the circuit.

If, then, we substitute for $R\dot{x}$ in equation $\frac{1}{2}\frac{dH}{d\dot{x}}$, we have as the general equation for the electromotive force in any mesh or cycle x,

$$\frac{d}{dt}\frac{dT}{d\dot{x}} + \tfrac{1}{2}\frac{dH}{d\dot{x}} = E.$$

This most important equation is Maxwell's general equation for determining the current $\dot{x}$ in any circuit when the dissipation function, and kinetic energy, and impressed electromotive force are known. We shall proceed to apply it to the solution of some network problems, in which the self and mutual induction of the branches is taken into account to determine the distribution of currents and combined resistance at any instant during the variable state.

§ 15. Consider, first, the case of a galvanometer with a coefficient of self-induction L and resistance G, and shunted by a shunt of resistance S, but wound so as to have no coefficient of self-induction, and let the shunt and galvanometer-coils be so far removed that there is no coefficient of mutual induction. This is the ordinary practical case.

Let a battery be joined up and let the battery and connections have a resistance B and electromotive force E (see fig. 16).

We have then a two-mesh network. Call the current in the galvanometer- and shunt mesh y and the current in the shunt and battery mesh $x+y$. Then the current through the galvonometer is y, the current through the shunt is x, and the current through the battery is $x+y$.

The dissipation function H is

$$\mathrm{B}\overline{x+y}^2+\mathrm{S}x^2+\mathrm{G}y^2=\mathrm{H},$$

which may be written

$$\overline{\mathrm{B}+\mathrm{S}}\ \overline{x+y}^2+\overline{\mathrm{G}+\mathrm{S}}\ y^2-2\mathrm{S}\,\overline{x+y}\,y=\mathrm{H};$$

and the electromagnetic energy is

$$\tfrac{1}{2}\mathrm{L}y^2=\mathrm{T}.$$

Hence, by the general equation,

$$\frac{d}{dt}\frac{d\mathrm{T}}{dy}+\tfrac{1}{2}\frac{d\mathrm{H}}{dy}=\mathrm{E},$$

we have the two cycle equations for the y and $x+y$ cycles,

$$\left.\begin{aligned}\frac{d}{dt}\mathrm{L}y+\overline{\mathrm{G}+\mathrm{S}}y-\mathrm{S}\overline{x+y}=0\\ \text{and}\\ \overline{\mathrm{B}+\mathrm{S}}\ \overline{x+y}-\mathrm{S}y=\mathrm{E},\end{aligned}\right\}$$

or

$$\left.\begin{aligned}\left(\mathrm{L}\frac{d}{dt}+\mathrm{G}\right)y-\mathrm{S}x=0,\\ \text{and}\\ \mathrm{B}y+\overline{\mathrm{B}+\mathrm{S}}\,x=\mathrm{E}.\end{aligned}\right\}$$

The solution of these for x and y is

$$y = \frac{\begin{vmatrix} E & B+S \\ 0 & -S \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through galvanometer,}$$

and

$$x = \frac{\begin{vmatrix} B & E \\ L\frac{d}{dt}+G & 0 \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through the shunt.}$$

Writing out this differential equation for y' we have,

$$\begin{vmatrix} L\frac{d}{dt} & -G \\ 0 & B+S \end{vmatrix} y + \begin{vmatrix} B & B+S \\ -G & S \end{vmatrix} y = \begin{vmatrix} E & -\overline{B+S} \\ 0 & S \end{vmatrix},$$

or $$\overline{B+S}\,L\frac{dy}{dt} + (BS+RG+SG)y = ES,$$

or $$\frac{dy}{dt} + \frac{BS+BG+SG}{(B+S)L}y = \frac{ES}{(B+S)L}.$$

The solution of this differential equation is

$$y = \frac{ES}{BG+GS+BS}\left(1-e^{-\frac{BG+GS+BS}{(B+S)L}t}\right).$$

This gives the value of the current through the galvanometer at any time, t, after starting the flow by making the connection with the battery.

When $t=0$, then $y=0$, and as t increases y increases, and finally, when $t=\infty$, $y=\frac{ES}{BG+GS+BS}$, or, as it may be written, $y=\frac{S}{G+S}\,\frac{E}{B+\frac{SG}{G+S}}$.

This last is the ordinary formula given for the current through a shunted galvanometer; but we see that when self-induction is taken into account, it is not until after an infinite time that the current rises to this value.

By the cycle equation, $By+\overline{B+S}x=E$: hence

$$x=\frac{E-By}{B+S}.$$

And if we write N for the factor $\left(1-e^{-\frac{BG+GS+BG}{(B+S)L}t}\right)$, then

$$x=\frac{E-\frac{EBSN}{BG+BS+SG}}{B+S}$$

$$=\frac{E(BG+BS+SG-BSN)}{(B+S)(BG+BS+SG)};$$

which gives the current through the shunt at any instant.

§ 16. Consider now the combined resistance of the galvanometer and shunt at any instant.

The self-induction of the galvanometer acts like a spurious resistance during the commencement of the current and drags out or prolongs the rise of current in the galvanometer-coils ; accordingly, during this period the combined resistance is a function of the time t from the commencement of the flow.

To calculate the combined resistance of galvanometer and shunt at any instant, we proceed as in the cases above exemplified. Form the cycle equations

$$(B+S)\overline{x+y}-Sy=E,$$

$$-S\overline{x+y}+\left(G+S+L\frac{d}{dt}\right)y=0.$$

Write down the determinant of these equations with the battery-circuit resistance put equal to zero, that is put $B=0$, and the combined resistance R required is the quotient of this determinant by its first minor, viz.

$$R=\frac{\begin{vmatrix} S & -S \\ -S & G+S+L\frac{d}{dt} \end{vmatrix}}{G+S+L\frac{d}{dt}}$$

or

$$R=\frac{\begin{vmatrix} S & 0 \\ -S & G+L\frac{d}{dt} \end{vmatrix}}{S+G+L\frac{d}{dt}}=\frac{S\left(G+L\frac{d}{dt}\right)}{S+\left(G+L\frac{d}{dt}\right)}.$$

We have now to see what is the meaning of $G+L\frac{d}{dt}$ as an operator in a determinant.

If we consider the formation of a current in a circuit of resistance R and coefficient of self-induction L by an electromotive force E, we have the equation for the current i

$$L\frac{di}{dt}+Ri=E.$$

Write thus

$$\left(L\frac{d}{dt}+R\right)i=E;$$

or, by notation of the calculus of operations,

$$i=E\left(L\frac{d}{dt}+R\right)^{-1}.$$

But now the solution of the above differential equation under the conditions $t=0$, $i=0$, and $t=\infty$, $i=\frac{E}{R}$, is

$$i=E\frac{\left(1-e^{-\frac{R}{L}t}\right)}{R}.$$

Comparing these two expressions for i together, we have

$$L\frac{d}{dt}+R=\frac{R}{\left(1-e^{-\frac{R}{L}t}\right)}.$$

Hence we may substitute in the expression for the combined resistance of galvanometer and shunt for $L\frac{d}{dt}+G$,

$$\frac{G}{\left(1-e^{-\frac{G}{L}t}\right)},$$

and we have as a result,

$$R=\frac{SG}{G+S\left(1-e^{-\frac{G}{L}t}\right)}.$$

We see that when $t=0$, $R=S$, and when $t=\infty$, $R=\frac{GS}{G+S}$.

Hence the result shows that at the first instant of starting a current through a shunted galvanometer, when the shunt has no self-induction and the galvanometer a considerable one, the galvanometer behaves as if it had a high spurious resistance, which in time dies away, allowing the total current,

after an infinite time, to be divided between the galvanometer and the shunt in the ratio of $\frac{S}{G+S}$ to $\frac{G}{G+S}$.

§ 17. We may apply the same methods to the examination of the case when the current sent through the shunted galvanometer is not generated by a source of constant electromotive force, but is a discharge from a condenser.

Let K (fig. 17) be a condenser connected up with a shunted galvanometer, so that when the key k is pressed a discharge passes through the galvanometer and shunt. Call the two cycles x and y. Let G be the galvanometer-resistance and S the shunt, and let L_1 and L_2 be their respective coefficients of self-induction ; the coefficient of mutual induction being zero.

Let q be the quantity of electricity in the condenser at any instant t. Counting the time from the instant of commencing the discharge, let C be the capacity of the condenser, and let q_1 and q_2 be the quantities of electricity which have, since the beginning of the epoch, flowed respectively through the galvanometer and the shunt.

If T be the energy function and F the dissipation function, we have, as above, the fundamental equations

$$2T = L_1 y^2 + L_2 (x-y)^2,$$

and

$$2F = Gy^2 + S(x-y)^2 ;$$

or

$$2T = L_1 y^2 + L_2 x^2 + L_2 y^2 - 2L_2 xy,$$
$$2F = Gy^2 + Sx^2 + Sy^2 - 2Sxy.$$

By the fundamental equation

$$\frac{d}{dt}\frac{dT}{dx} + \frac{dF}{dx} = e.$$

For e we must write $\frac{q}{C}$.

Writing, then, the cycle equations, we have

$$\frac{d}{dt}(L_2 x - L_2 y) + Sx - Sy = \frac{q}{C}.$$

$$\frac{d}{dt}(L_1 y + L_2 y - L_2 x) + Gy + Sy - Sx = 0 ;$$

from which we deduce easily

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C},$$

and

$$L_2\frac{d}{dt}(x-y)+S(x-y)=\frac{q}{C};$$

or

$$L_1\frac{dy}{dt}-L_2\frac{d}{dt}(x-y)=S(x-y)-Gy.$$

Now y and $x-y$ represent the strengths of the currents flowing through the galvanometer and the shunt at any instant.

If we integrate both sides of the equation from 0 to ∞, we have

$$\left[L_1y-L_2(x-y)\right]_0^\infty=S\int_0^\infty\overline{x-y}dt-G\int_0^\infty ydt.$$

Now the left-hand side of the equation is zero because quantities of the form of Ly represent the number of lines of force which are added into the circuit of the galvanometer, and the discharge may be divided into two parts, during one of which lines of force are being added to, and in the other of which subtracted from, the circuits of the galvanometer and shunt; and the sum of these is zero. Again, $\int_0^\infty(x-y)dt$ and $\int_0^\infty ydt$ represent the whole quantities q_2 and q_1 of electricity which have flowed respectively through the galvanometer and the shunt. Hence we arrive at the conclusion that

$$Sq_2-Gq_1=0,$$

or

$$\frac{G}{S}=\frac{q_2}{q_1};$$

that is, the total quantity of the discharge is divided between the two circuits inversely as their resistances. We see therefore that self-induction does not affect the ratio of division of a discharge in a divided circuit, provided that no external work, such as the moving of magnets or circuits conveying currents, absorbs current energy. Hence, if a ballistic galvanometer is shunted and a discharge sent through it, if the needle has sufficient moment of inertia and the discharge is sufficiently short, so that the needle has not perceptibly moved from its position before the discharge is over, then the whole quantity of electricity is divided between the galvanometer and the shunt in the inverse ratio of their resistances.

§ 18. To complete the solution we have to calculate the current flowing through the galvanometer and shunt at any instant.

Taking the two cycle equations

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}, \quad \ldots \quad \text{(i.)}$$

and

$$L_2 \frac{d}{dt}\overline{x-y} + S\overline{x-y} = \frac{q}{C}, \quad \ldots \quad \text{(ii.)}$$

we get

$$L_1 \frac{dy}{dt} = \frac{q}{C} - Gy,$$

and

$$L_1 L_2 \frac{dx}{dt} = (L_1 + L_2)\frac{q}{C} - L_1 S\overline{x-y} - L_2 Gy$$

$$= (L_1 + L_2)\frac{q}{C} - L_1 Sx + (L_1 S - L_2 G)y. \quad \text{(iii.)}$$

Differentiate this last equation with regard to t and eliminate $\frac{dy}{dt}$ by the help of the equation above it, and we arrive at

$$L_1 L_2 \frac{d^2x}{dt^2} = -L_1 S\frac{dx}{dt} + \frac{L_1 S - L_2 G}{L_1}\left(\frac{q}{C} - Gy\right).$$

Eliminating y between the last and equation (iii.) and reducing, we arrive at

$$L_1 L_2 \frac{d^2x}{dt^2} + (L_1 S + L_2 G)\frac{dx}{dt} + GSx = (G+S)\frac{q}{C};$$

but now $x = -\frac{dq}{dt}$. Making this substitution we have

$$L_1 L_2 C \frac{d^3q}{dt^3} + C(L_1 S + L_2 G)\frac{d^2q}{dt^2} + CGS\frac{dq}{dt} + (G+S)q = 0, \quad \text{(iv.)}$$

an interesting equation, the solution of which gives us the quantity of electricity in the condenser at any instant, t, after starting the discharge. According to the equation above

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}.$$

This equation gives us a value of y or the current through the galvanometer at any instant when we know q, or the

quantity left in the condenser at that instant. The above may be written

$$y = \left(L_1\frac{d}{dt} + G\right)^{-1}\frac{q}{C},$$

and the final equation (iv.) may be written

$$\frac{q}{C} = \left(L_1L_2C^2\frac{d^3}{dt^3} + (L_1S + L_2G)C^2\frac{d^2}{dt^2} + C^2GS\frac{d}{dt} + C(G+S)\right)^{-1}0;$$

and accordingly we have the following equation for the value of y at any instant

$$y = \left(L_1\frac{d}{dt} + G\right)^{-1}\left(L_1L_2C^2\frac{d^3}{dt^3} + (L_1S + L_2G)C^2\frac{d^2}{dt^2} + C^2GS\frac{d}{dt} + C(G+\right.$$

which may be written

$$y = \left(L_1^2L_2C^2\frac{d^4}{dt^4} + (L_1^2SC^2 + 2L_1L_2C^2G)\frac{d^3}{dt^3} + (2L_1C^2GS + L_2G^2C\right.$$
$$\left. + (G^2 + SCL_1 + C^2G^2S)\frac{d}{dt} + GC(G+S)\right)^{-1}0.$$

This linear differential equation is solved when we know the roots of the auxiliary biquadratic; and according as they are all real or partly imaginary, so will be the nature of the solution.

If the roots are all real the solution is a sum of exponentials, whose total value first increases and then dies away as t increases, indicating that the discharge produces a wave of electricity through the galvanometer always in one direction; but if two or all of the roots of the auxiliary biquadratic are unreal, it indicates as the form of solution a function of sines and cosines which will have periodic values, and points to the fact that the discharge is a series of alternations. The general case, when both the galvanometer and shunt have coefficients of self-induction, when treated to determine the conditions for an oscillating discharge, leads to an expression of considerable complexity and not much practical use. The reduced case, in which the galvanometer is wound to have self-induction and the shunt so as to have no coefficient of self-induction, is, however, a practical case, and can be treated without much difficulty.

Taking the differential equation for q, equation (iv.), and

writing in it $L_2=0$, we have

$$CL_1S\frac{d^2q}{dt^2}+CGS\frac{dq}{dt}+(G+S)\,q=0,$$

or

$$\frac{S}{G+S}\frac{d^2q}{dt^2}+\frac{GS}{L_1(G+S)}\frac{dq}{dt}+\frac{q}{CL_1}=0.$$

The discharge will be oscillatory if the auxiliary quadratic

$$\frac{S}{G+S}m^2+\frac{GS}{L_1(G+S)}m+\frac{1}{CL_1}=0$$

has unreal or imaginary roots.

Solving it we have

$$m^2+\frac{Gm}{L_1}+\frac{G^2}{4L_1^2}=\frac{G^2}{4L_1^2}-\frac{G+S}{SCL_1},$$

or

$$\left(m+\frac{G}{2L_1}\right)=\pm\frac{\sqrt{G^2S^2C^2-4G+SL_1SC}}{2L_1SC.}$$

Hence, for the roots to be imaginary,

$4L_1\,G+S\,SC$ must be greater than $G^2S^2C^2$,

or

$$\frac{4L^1}{C}>\frac{G\,.\,GS}{G+S}.$$

If this relation holds good, then the discharge is oscillatory in the condenser; and accordingly we see that to prevent electrical oscillation in the galvanometer circuit, the product of resistance of the galvanometer and combined resistance of galvanometer and shunt must be equal to or greater than four times the self-induction of the galvanometer divided by the capacity of the condenser.

We may write the solution of the quadratic above,

$$m=-\frac{G}{2L_1}\pm\sqrt{-1}\sqrt{\frac{G+S}{\frac{S}{L_1C}-\frac{G^2}{4L_1^2}}}$$
$$=-\alpha\pm\sqrt{-1}\beta,$$

where

$$\alpha=\frac{G}{2L_1}\quad\text{and}\quad\beta=\sqrt{\frac{G+S}{\frac{S}{L_1C}-\frac{G^2}{4L_1^2}}};$$

and accordingly when β is real, that is when

$$\frac{\frac{G+S}{S}}{L_1C} \text{ is } > \frac{G^2}{4L_1^2},$$

we have, for solution of equation,

$$q = Ae^{-at}\cos\beta t + \beta e^{-at}\sin\beta t.$$

When $t=0$, $q=Q=$the original charge of the condenser, and $\frac{dq}{dt}=0$ when $t=0$;

therefore $\qquad Q=A \quad \text{and} \quad Q\frac{a}{\beta}=B;$

and $$q = Qe^{-at}\left(\cos\beta t + \frac{a}{\beta}\sin\beta t\right).$$

Having now the value of the quantity of electricity left in the condenser at any instant, we can find easily, from the cycle equation (i.), the value of the current through the galvanometer. For

$$L_1\frac{dy}{dt} + Gy = \frac{q}{C},$$

or

$$\frac{dy}{dt} + \frac{G}{L}y = \frac{q}{CL};$$

$$\therefore \quad y = e^{-\int\frac{G}{L}dt}\left\{C' + \int e^{\int\frac{G}{L}dt}\frac{q}{CL}\,dt\right\},$$

and the constant C' is determined by the condition $y=0$ when $t=0$.

Substituting the value of q above, we have

$$a = \frac{G}{2L},$$

$$y = e^{-2at}\left\{C' + \frac{1}{\beta}\frac{Q}{CL}\int e^{at}(\beta\cos\beta t + a\sin\beta t)\right\}$$

$$= e^{-2at}\left\{C' + \frac{Q}{\beta CL}e^{at}\sin\beta t\right\};$$

but $C'=0$,

$$\therefore \quad y = \frac{Q}{\beta CL_1}e^{-at}\sin\beta t;$$

and since

$$a=\frac{G}{2L_1} \quad \text{and} \quad \beta=\sqrt{\frac{G+S}{SL_1C}-\frac{G^2}{4L_1^2}},$$

$$y=\frac{Q}{\sqrt{\frac{G+S}{S}CL_1-\frac{G^2C^2}{4}}}e^{-\frac{G}{2L_1}t}\sin\left(\sqrt{\frac{G+S}{SL_1C}-\frac{G^2}{4L_1^2}}\right)t,$$

which gives the value of the instantaneous current in the galvanometer-circuit at any instant t after starting a discharge from a condenser of capacity C and original quantity Q through a shunted galvanometer, the shunt being wound without self-induction, and the galvanometer having a coefficient of self-induction L_1.

§ 19. Two concluding examples of this method of treating network problems will now be given, which are in Professor Clerk Maxwell's own words*.

Theorem.—To compare the induction between one pair of coils and any other two.

Let α, β, γ, δ be four coils of wire.

It is required to compare the mutual induction of α and γ with that of β and δ.

Join up α and β coils in series with a galvanometer, and join up γ and δ coils in multiple arc with a battery, as shown in fig. 18.

Place the coils in position.

* In the May term 1879, Professor Clerk Maxwell lectured at Cambridge on Electromagnetism, and in the two last lectures of the Course he gave this method of obtaining the equation for the currents in a network of conduction. In the last lecture of all he applied the method to cases in which self and mutual induction was taken into account, and gave the two illustrations in § 19. At the conclusion of this lecture he had ended his professorial duties for the term, and a melancholy interest attaches to the subject which occupied his mind on the last occasion on which, unconsciously to himself or his pupils, he was to perform them. Those who enjoyed even for a brief period the privilege of being taught by him, ever cherish a vivid remembrance of the intellectual treat afforded by Professor Maxwell's lecture-teaching, and the profound suggestiveness and interest of it.

The two examples in § 19 and § 20 are taken from my notes of Prof. Maxwell's lectures, with some little alterations to make them clearer.

Let R and S be resistances of the primaries γ and δ; let Q be resistance of the two secondaries and of the galvanometer.

Let L_1, L_2, N_1, N_2 be the coefficients of self-induction of the coils, and M_1, M_2 the coefficients of mutual induction of α and γ, β and δ. Let Γ be the coefficient of self-induction of the galvanometer.

Call x the cycle current of γ, y that of δ, and z that of the circuit formed of α, β, and the galvanometer.

The kinetic energy T of the system is

$$2T = x^2N_1 + 2xzM_1 + z^2(L_1 + L_2 + \Gamma) + y^2N_2 + 2yzM_2,$$

and the dissipation function F is

$$2F = x^2R + y^2S + z^2Q.$$

Then, by the formula

$$E = \frac{d}{dt}\frac{dT}{dx} + \frac{dF}{dx},$$

$$\dot{x}N_1 + \dot{z}M_1 + xR = E.$$

$$\dot{y}N_2 + \dot{z}M_2 + yS = -E.$$

$$\dot{x}M_1 + \dot{y}M_2 + \dot{z}(L_1 + L_2 + \Gamma) + zQ = 0.$$

Now $x-y$ is the current through the battery; hence if we put $x+y$ for x in the above, we shall get x as the battery-current. Hence, making the change, we have

$$(\dot{x}+\dot{y})N_1 + \dot{z}M_1 + \overline{x+y}R = E, \quad \ldots\ldots \quad \text{(i.)}$$

$$\dot{y}N_2 \qquad + \dot{z}M_2 + yS = -E, \quad \ldots\ldots \quad \text{(ii.)}$$

$$(\dot{x}+\dot{y})M_1 + \dot{y}_1M_2 + \dot{z}(L_1 + L_2 + \Gamma) + zQ = 0; \quad \text{(iii.)}$$

add equations (i.) and (ii.) and arrange, putting n for $\frac{d}{dt}$,

$$(N_1n + R)x + (N_1n + N_2n + R + S)y + (M_1 + M_2)nz = 0,$$

$$M_1nx + (M_1n + M_2n)y + \{(L_1 + L_2 + \Gamma)n + Q\}z = 0.$$

Eliminating y, we have

$$\{n(N_1n + R)(M_1 + M_2) - M_1n(N_1n + R + S)\}x$$

$$+ \{(M_1 + M_2)^2n^2 - \{(L_1 + L_2 + \Gamma)n + Q\}\{N_1n + N_2n + R + S\}\}z = 0.$$

Hence we get

$$z = \frac{\{n^2(M_2N_1 - M_1N_2) - n(M_1S - M_2R)\}x}{\text{a denominator which does not concern us}}.$$

If matters are so arranged that $z=0$, or the galvanometer shows no current,

$$(M_2N_1-M_1N_2)\frac{dx}{dt}=(M_1S-M_2R);$$

hence if there is no "kick" on the galvanometer on making the current, then

$$\frac{M_1}{M_2}=\frac{R}{S}.$$

§ 20. *Theorem.*—To determine the capacity of a condenser by means of a Wheatstone's bridge (fig. 19).

Let α, β, γ, δ be the four points of a Wheatstone's bridge; and let the branch between α and β be interrupted at a b, and a Leyden jar or condenser inserted provided with some rapid commutator, such as a tuning-fork, so that whilst the outside of the jar is kept permanently attached to β, the inside is alternately joined to a and b.

If a tuning-fork is used and its prongs have small metal styles which just come down to the surface of the mercury in two little cups, when the fork vibrates, as the prongs come together, the upper point dips in; and as they separate, the lower one dips in; hence the shank of the fork is alternately connected with one and the other cup. The interval between the time of connection being exactly half the time of a complete oscillation of the fork.

Now let the meshes of the network be called $x+z$, z, and y; then x is the current through the galvanometer, and y is the current through the battery. When the arrangement is made as in the diagram, and the fork set vibrating, the vibrating fork and the condenser act together like a resistance, and let through so much electricity per second.

Now, as the condenser gets its charge by electricity flowing into it, it builds up an opposing electromotive force in the z circuit which at any instant is equal to the value of $\frac{\int z dt}{K}$, where K is the capacity of the jar, the integral being integrated from the instant when the charging commences up to the instant considered. Now, if the fork makes n vibrations a second when the steady state is set up, the current z which flows into

the jar has a mean value z; and therefore $\frac{z}{n\mathrm{K}}$ is the opposing electromotive force in that branch.

Accordingly, the condenser and associated commutator behave like a voltameter inserted in the branch $\alpha\beta$, or like a resistance with a counter electromotive force in it. Only such a combined jar and fork differs from an ordinary metallic resistance in this, that its apparent resistance is not constant, but depends on two things, the speed of commutation or charge and recharge, and the capacity of the condenser; whilst the counter electromotive force depends on the current z, and, being represented by $\frac{z}{n\mathrm{K}}$, is dependent not only on n and K, but also on the values of all the other resistances in the branches. In the first place, we require an expression for the electromotive force charging the condenser. Let the difference of potential between a and b be called e. Then consider the network formed by the five conductors R, S, Q, G, and B with the electromotive force in the branch B; write down the network equations for this z mesh network.

$$(\mathrm{B}+\mathrm{R}+\mathrm{S})y-\mathrm{S}(x+z) \quad =\mathrm{E},$$

$$-\mathrm{S}y+(\mathrm{Q}+\mathrm{S}+\mathrm{G})(x+z)=0.$$

Hence

$$y=\frac{\mathrm{E}(\mathrm{Q}+\mathrm{S}+\mathrm{G})}{\delta};$$

and

$$x+z=\frac{\mathrm{ES}}{\delta},$$

where $\delta=$ the determinant

$$\begin{vmatrix} \mathrm{B}+\mathrm{R}+\mathrm{S}, & -\mathrm{S} \\ -\mathrm{S}, & \mathrm{Q}+\mathrm{S}+\mathrm{G} \end{vmatrix}$$

which is

$$\mathrm{S}(\mathrm{Q}+\mathrm{G})+(\mathrm{R}+\mathrm{B})(\mathrm{Q}+\mathrm{S}+\mathrm{G}).$$

Now the difference of potential e between a and b when the condenser is just beginning to be charged is

$$\mathrm{G}(x+z)+\mathrm{R}y=e;$$

$$\therefore\ e=\frac{\mathrm{ESG}}{\delta}+\frac{\mathrm{ER}(\mathrm{Q}+\mathrm{S}+\mathrm{G})}{\delta}$$

$$=\mathrm{E}\frac{\mathrm{SG}+\mathrm{R}(\mathrm{Q}+\mathrm{S}+\mathrm{G})}{\mathrm{S}(\mathrm{Q}+\mathrm{G})+(\mathrm{R}+\mathrm{B})(\mathrm{Q}+\mathrm{S}+\mathrm{G})}=\frac{\begin{vmatrix}-\mathrm{G}, & -\mathrm{R}\\ \mathrm{Q}+\mathrm{S}+\mathrm{G}, & -\mathrm{S}\end{vmatrix}}{\mathrm{S}(\mathrm{Q}+\mathrm{G})+(\mathrm{R}+\mathrm{B})(\mathrm{Q}+\mathrm{S}+\mathrm{G})}$$

or

$$e=\mathrm{E}\frac{\begin{vmatrix}-\mathrm{G}, & -\mathrm{R}\\ \mathrm{Q}+\mathrm{S}+\mathrm{G}, & -\mathrm{S}\end{vmatrix}}{\delta}.$$

Now if the electromotive force e be employed n times in a second to charge a jar of capacity K, the average current flowing into the jar is $n\mathrm{K}e=z$.

Now to find z we have to consider the distribution of currents when the fork or commutator is in operation, and the condenser allowing a flow of electricity to take place through it.

Let P be the resistance which could equivalently replace the jar and fork—that is, would allow an equal quantity of electricity to pass per second; then, since $\frac{z}{n\mathrm{K}}$ is the opposing electromotive force in this branch, we have the following equation for the three cycles x, $x+z$, and y:—

$$-\mathrm{S}x+(\mathrm{R}+\mathrm{S}+\mathrm{B})y-(\mathrm{R}+\mathrm{S})z=\mathrm{E},$$

$$-\mathrm{G}x-\mathrm{R}y+(\mathrm{P}+\mathrm{R})z=-\frac{z}{n\mathrm{K}},$$

$$(\mathrm{Q}+\mathrm{S}+\mathrm{G})x-\mathrm{S}y+(\mathrm{Q}+\mathrm{S})z=0.$$

Now let Δ stand for the determinant

$$\begin{vmatrix}-\mathrm{S}, & \mathrm{R}+\mathrm{S}+\mathrm{B}, & -(\mathrm{R}+\mathrm{S})\\ -\mathrm{G}, & -\mathrm{R}, & \mathrm{P}+\frac{1}{n\mathrm{K}}+\mathrm{R}\\ \mathrm{Q}+\mathrm{S}+\mathrm{G}, & -\mathrm{S}, & \mathrm{Q}+\mathrm{S}\end{vmatrix}$$

Then the solution of the above equations for z and x are

$$z=\frac{\mathrm{E}\begin{vmatrix}-\mathrm{G}, & -\mathrm{R}\\ \mathrm{Q}+\mathrm{S}+\mathrm{G}, & -\mathrm{S}\end{vmatrix}}{\Delta},$$

and

$$x=\frac{\mathrm{E}\begin{vmatrix}-\mathrm{R}, & \mathrm{P}+\frac{1}{n\mathrm{K}}+\mathrm{R}\\ -\mathrm{S}, & \mathrm{Q}+\mathrm{S}\end{vmatrix}}{\Delta}.$$

z is the average current flowing through the condenser, and x is the current through the galvanometer. Now let the resistances R, S, and Q be so varied that the current through the galvanometer is zero, then $x=0$; and therefore

$$\begin{vmatrix} -R, & P+\frac{1}{nK}+R \\ -S, & Q+S \end{vmatrix} = 0,$$

or

$$R(Q+S) = S\left(P+\frac{1}{nK}+R\right),$$

or

$$\frac{RQ}{S} = P+\frac{1}{nK}.$$

Now insert this value for $P+\frac{1}{nK}$ in the determinant Δ above and calculate its value, and we arrive at the expression

$$\Delta = \frac{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}}{S}.$$

We have now, by substitution of this value of Δ in the value obtained above for z, an expression for the value of the average current through the condenser when the bridge is balanced, and it is

$$z = \frac{ES\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}}.$$

Equating this to the other value for z, namely,

$$z = nKe = nKE\frac{\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{S(Q+G)+(R+B)\{Q+S+G\}},$$

we have

$$nK = \frac{S\{S(Q+G)+(R+B)(Q+S+G)\}}{\{B(Q+S)+Q(R+S)\}\{G(R+S)+R(Q+S)\}},$$

which gives us a value for nK in terms of

$$B,\ Q,\ R,\ S,\ G.$$

Now it is interesting to note that we may otherwise write the above expression for nK,

$$\frac{1}{nK} = \frac{\Delta}{\delta},$$

where Δ is the determinant,

$$\begin{vmatrix} R+B+S, & -S, & -\overline{R+S} \\ -S, & Q+S+G, & R\left(\frac{Q}{S}+1\right) \\ -R, & -G, & Q+S \end{vmatrix},$$

and δ is its first minor,

$$\begin{vmatrix} R+B+S, & -S \\ -S, & Q+S+G \end{vmatrix},$$

and $\frac{1}{n\mathrm{K}}$ is of the dimensions of a resistance.

The value for $n\mathrm{K}$ writes out by a simple transformation into another form,

$$n\mathrm{K} = \frac{S\left\{1-\frac{S^2}{(Q+S+G)(R+B+S)}\right\}}{RQ\left\{1+\frac{SB}{Q(R+B+S)}\right\}\left\{1+\frac{SG}{R(Q+S+G)}\right\}};$$

which is the form in which it is given by Prof. J. J. Thomson in his paper, and quoted by Mr. R. T. Glazebrook in his memoir on a Method of Measuring the Capacity of a Condenser*.

The above examples are amply sufficient to exemplify this method of treating problems in networks of conductors, and show how it enables calculations to be made with considerable ease, not only of the distribution of currents and potentials, but of the resistances between any points on a network, the branches of which consist either of simple resistances or of wires having self- and mutual induction with other branches, or of electromagnets, or condensers associated with appropriate commutators.

* This method of Maxwell's, of obtaining the capacity of a condenser has been practically employed, with most excellent results, by Mr. R. T. Glazebrook, F.R.S.; and the full details of the tests to which he subjected the method are given in his paper in the 'Proceedings of the Physical Society,' vol. vi. part iii. p. 204 (June 28, 1884). [Phil. Mag. for August 1884, p. 98.]

XXVIII. *On the Generation of Electric Currents by Sulphur Cells.* *By* Shelford Bidwell, *M.A., LL.B.**

In a communication recently made to the Physical Society† I ventured the suggestion that the electric conduction of selenium, when prepared in such a form as to be sensitive to light, was, in the literal sense of the term, electrolytic. Selenium itself indeed could hardly be supposed to be an electrolyte; but it was pointed out that when selenium was "annealed" in contact with metallic electrodes, metallic selenides would, at least in most cases, be formed in sufficient quantity to account for the electrolytic phenomena observed; and that even if this were not the case when the electrodes consisted of such a metal as platinum, yet the necessary metallic element might possibly be found in the lead, iron, and arsenic which are contained as impurities in ordinary commercial selenium. Little direct evidence was offered in support of this view; but it was shown that sulphur, when mixed with a certain proportion of sulphide of silver and arranged in the form of a "cell" with silver electrodes, exhibited many of the properties of crystalline selenium, especially that of having its electrical resistance temporarily diminished under the influence of light. Analogy therefore tended to confirm the opinion which I had been led to entertain.

While observing the secondary or polarization-currents which are generated by sulphur cells (as by those made with selenium), after being disconnected from a battery, certain effects were noticed which seemed to indicate that when the electrodes consisted of two different metals, a sulphur cell might be capable of originating and maintaining an independent or primary current. Experiments were therefore made with the object of investigating this point; and the present paper contains an account of the results obtained. I have hardly attempted to connect them together by any complete theory: of some of them, indeed, I can offer no explanation whatever; others appear to be in direct opposition to what might have been expected. But, so far as I have been able

* Read June 27, 1885.

† Phil. Mag. August 1885.

to ascertain, they are entirely novel, and of sufficient interest to be worthy of record.

(1) A slip of mica was wound with two parallel wires of silver and copper 1 millim. apart, and melted sulphur containing a small quantity of precipitated sulphides of silver and copper was spread over one surface. It is not known what proportion of the sulphides was contained in the mixture, because the bulk of them sank to the bottom of the crucible in which the sulphur was melted. When cold, the cell was connected with a reflecting galvanometer, and was found to generate a small but steady current, indicated by a deflection of about 20 scale-divisions. When the connections of the cell were reversed, the current was reversed. It was therefore not due to any thermo-effect in the circuit. The direction of the current was from silver to copper through sulphur: its strength was *diminished* by exposure to light, and *increased* by rise of temperature. Connecting the free ends of the silver and copper wires and heating the junction, it was found that the thermo-current thus produced passed (as usual) from copper to silver through the junction. The increase of current by heat was therefore not to be accounted for by thermo-electric action, for that would produce the opposite effect.

The action is almost certainly of the same nature as that which occurs in an ordinary voltaic cell.

(2) It appeared desirable to construct a cell which, though unsuitable for experiments with light, would have a much smaller resistance than one of the form last described. A plate of copper 3 centim. square was heated, and upon it was spread a mixture consisting of 5 parts of sulphur and 1 part of sulphide of copper. A plate of silver previously heated was then laid on the melted mixture and the two plates squeezed together, thus forming a sandwich-like cell. The thickness of the copper plate was 2·75 millim., of the silver plate ·60 millim., and of the completed cell 3·65 millim.; the thickness of the layer of sulphur was therefore ·3 millim. When this cell (after cooling) was connected with the galvanometer, the spot of light was violently deflected off the scale. Dr. Fleming was kind enough to make a very accurate measurement of its electromotive force by comparison with one of his standard Daniell cells. It was found to be ·0712 volt, and its internal

resistance was 6537 ohms. As in the case of the former cell, the direction of the current is from silver to copper; and there can be no doubt that it is of a voltaic nature. At the time of writing, the cell has been in existence nearly seven weeks, and it is now, I believe, quite as powerful as at first.

(3) It was thought that the internal resistance might be further reduced by adding a larger proportion of sulphide to the sulphur. Another cell was therefore constructed similar in all respects to that last described, except that the sulphur and copper sulphide were mixed in equal proportions. Its internal resistance was enormously lower, being only 13 ohms, but its E.M.F. was also lower, being ·0071 volt.

(4) A layer of precipitated sulphide of copper was placed between plates of copper and silver which were squeezed together in a screw-press. The resistance of this arrangement was a small fraction of an ohm; but when connected with the galvanometer, it gave no indication whatever of a current. It seems, therefore, that a certain amount of free sulphur is necessary for the generation of an electromotive force by cells containing copper sulphide.

(5) Two parts of copper sulphide were mixed with one of sublimed sulphur, and the powder was compressed between plates of copper and silver. This cell gave a very small current, indicated by a galvanometer-deflection of 2 or 3 scale-divisions. The deflection was reversed as often as the connections with the binding-screws of the cell were reversed; and the existence of a real, though very small, electromotive force was undoubted. The internal resistance was ·088 ohm.

(6) The last-mentioned cell was taken to pieces and remade after the addition of about an equal part of sublimed sulphur to the mixture of sulphur and sulphide. Its resistance was now found to be many megohms, yet it produced a larger galvanometer-deflection than before.

(7) Once more the cell was taken to pieces and a little more sulphide added. When remade, its resistance was at first about 2700 ohms; but it varied considerably. It produced a galvanometer-deflection of about 100 divisions, which in a few minutes increased in a somewhat irregular manner to 250 divisions. It was then disconnected from the galvanometer, and, when again connected after an interval of six

hours, it deflected the spot of light off the scale. Shunting the galvanometer (the resistance of which was 3483 ohms) with a coil of 300 ohms, the deflection amounted to 130 divisions; and on the following day with the same shunt, the deflection was at first about 250 divisions, rapidly diminishing, however, when the circuit was closed.

(8) A cell was made by compressing precipitated silver sulphide unmixed with any free sulphur between plates of silver and copper. When connected with the galvanometer, this cell produced a deflection which, with a shunt of 35 ohms, exceeded 400 divisions. But in this case the *silver* was the negative plate, the direction of the current being from copper through sulphide to silver. The E.M.F. was less than that of the cell described in (2).

(9) Another cell was made in the same manner as that described in (2); but the sulphur was mixed with sulphide of silver instead of sulphide of copper. This gave a strong current in the same direction as that produced when sulphide of copper was used, and opposite to that generated by the cell containing silver sulphide without free sulphur.

(10) Pure sulphur was melted on a clean plate of copper, and, when just liquid, a warmed plate of silver was laid upon it and pressed down with a weight until cold. This cell gave a strong current from silver through sulphur to copper. Sulphides were of course formed during the process of construction.

(11) A melted mixture of sulphur and silver sulphide was spread upon a copper plate, and a plate of silver pressed upon it. When cold, the silver plate was split off with a knife, and a piece of silver-leaf, sufficiently thin to appear blue by transmitted light, was attached (by rubbing) to the exposed surface of the mixture. As before, there was a comparatively strong current from the silver to the copper. The silvered surface was then exposed to the light of burning magnesium wire, and the immediate movement of the spot of light through 50 scale-divisions indicated a *diminution* of the current. When the magnesium was extinguished, the current at once increased to its original strength. When a nearly red-hot brass rod was held at a distance of 3 centims. from the silvered surface, the current slowly *increased* in strength; and when the hot

rod was removed, the current was again slowly diminished. The effects both of light and of heat were verified by many repetitions of the experiments. These results which, so far as regards the effect of light, were unexpected, are of the same character as those described in (1). I have elsewhere* given strong reasons for believing that the combination of sulphur with silver is assisted by the influence of light. If this is so, it is certainly a remarkable fact that increased corrosion of the silver electrode should be accompanied by diminution of the current.

Two days afterwards the silver-leaf had become much discoloured, and was in some parts quite black. When connected with the galvanometer the cell gave a current of nearly the same strength as before; but now it was found to be slightly increased by light as well as by heat; and it is probable that the light as such exerted no influence whatever, the observed effect being really due to the incidental rise of temperature.

(12) The silver-leaf was scraped off, and the surface of the sulphur mixture having been cleaned from all visible traces of free silver with fine emery-cloth, a piece of thin gold-leaf was pressed upon it. It was found very difficult to make it adhere satisfactorily. The cell, when connected with the galvanometer, gave no indication whatever of a current. When the cell described in (2) was also inserted in the circuit, the spot of light was deflected, showing that the first cell was quite able to conduct electricity, and that its failure to originate a current was not owing to bad contact between the gold-leaf and the sulphur mixture.

(13) A cell containing a mixture of sulphur and silver sulphide between plates of silver and iron was found to have an E.M.F. of ·023 volt, or about one third of that of the silver-copper cell described in (2). The direction of the current was the same.

(14) A cell was made by melting sublimed sulphur upon a plate of copper and pressing a plate of iron upon the melted sulphur. On connecting it (when cold) with the galvanometer, there was no indication of any current. Nevertheless the cell was found to conduct electricity very well when a

* Phil. Mag. August 1885.

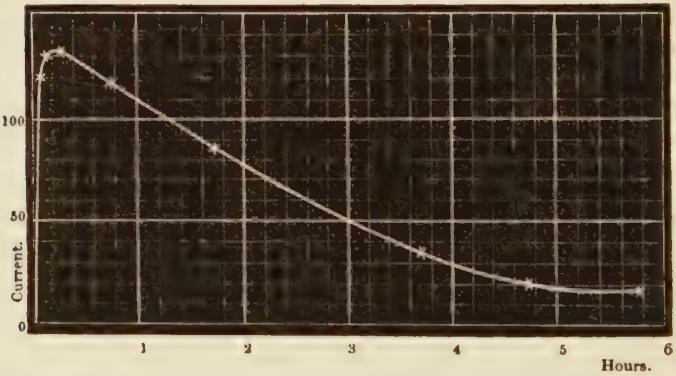
battery was also placed in the circuit. Sulphide of copper was of course formed by the action of the hot sulphur upon the copper.

(15) A melted mixture of 5 parts of sulphur with 1 part of copper sulphide was pressed between plates of silver and iron, and cooled. When first made, this cell generated a sufficiently strong current to deflect the galvanometer-needle as far as the stop would allow; but two hours later, when an attempt was made to measure the E.M.F., it was found to have almost completely disappeared, being less than a hundredth of that of the cell described in (2). The E.M.F. was temporarily restored by connecting the iron and silver plates with the positive and negative terminals of a battery of ten Leclanche cells for a few minutes. The result was of course merely a polarization-current.

(16) Two silver wires A, B were imbedded in a fused mass consisting of equal parts of sulphur and copper sulphide. When cold, the wire A was connected with the carbon pole of a battery of ten Leclanché cells, and the wire B with the zinc pole. After the current had passed for about a second, the cell was detached from the battery and connected with the galvanometer. A current was at once indicated in the direction A B, *i. e.* in the *same* direction as that of the battery-current which had been caused to pass through the cell. This experiment was repeated many times and on different days, with the same result*. A period of some seconds necessarily elapsed between the separation of the cell from the battery and its attachment to the galvanometer. In order to render this interval as short as possible, the apparatus was so arranged that by depressing a key the transfer could be effected in a small fraction of a second. It was then found that the first effect of the transfer was a strong *momentary* current in the direction B A, which was immediately followed by the more permanent current previously observed in the direction A B. With the view of retarding the first effect, a battery of two Leclanché cells was used, and the connection was made for a longer period. After the battery had been connected for one minute, the cell was transferred to the galvanometer

* The experiments described in the remainder of this paragraph were made after the paper was read.

by means of the key, and the swing of the spot of light through 280 scale-divisions again indicated a current in the direction B A. Though this was not, as before, a current of only momentary duration, it rapidly decreased in strength, becoming zero in almost exactly 30 seconds. After zero was passed, a current was at once set up in the opposite direction A B, which in 30 seconds produced a deflection of 70 divisions, increasing to a maximum of 131 divisions in 10 minutes. The current then slowly diminished; and in $5\frac{3}{4}$ hours after the commencement of the experiment the deflection had fallen to 17 scale-divisions. At this point the observations were discontinued. In the accompanying curve the abscissæ represent the time in hours, and the ordinates the current in scale-divisions. The current of 30 seconds' duration in the direction B A is not represented.



On another occasion, when the two Leclanché cells had been connected for 3 minutes, the secondary current from B to A decreased still more slowly than in the last-mentioned experiment, vanishing in $3\frac{1}{2}$ minutes. The current which followed in the direction A B attained its maximum, indicated by 155 divisions, in $6\frac{1}{2}$ minutes, and then steadily decreased for nearly 4 hours, when it again became zero. The spot of light did not, however, remain stationary, but moved steadily on, indicating a second reversal of the current. In one hour after zero had thus been crossed for the second time, the galvanometer-deflection was 12 scale-divisions; and $4\frac{1}{2}$ hours later, when the last observation was made, it had increased to 20 divisions. In both these experiments the galvanometer

(of which the resistance was nearly 3500 ohms) was shunted with a coil of 100 ohms.

These alternating currents are probably of the same nature as those which Faraday found to be generated when copper and silver, or two pieces of copper, or two pieces of silver, were immersed in a solution of potassium sulphide*. The wires are alternately protected from the action of the free sulphur by an investing coat of sulphide.

Summary.

Plates of silver and copper imbedded in a mixture of sulphur with sulphides of copper or silver constitute a cell which at the ordinary temperature is capable of generating and maintaining a constant current, the silver being the positive element. Such a cell, in which the mixture consisted of 5 parts of sulphur with 1 of copper sulphide between plates 3 centim. square and ·3 millim. apart, had an E.M.F. of ·0712 volt and an internal resistance of 6537 ohms.

If the proportion of copper sulphide to sulphur is increased, the internal resistance of the cell is diminished; but its E.M.F. is also diminished.

A cell containing copper sulphide unmixed with free sulphur fails to produce any appreciable current.

A cell containing silver sulphide only generates a current in the opposite direction to that produced when the sulphide is mixed with free sulphur.

Copper used in conjunction with iron or gold gives no current whatever at the ordinary temperature.

The current generated by a silver-copper cell containing free sulphur mixed with sulphide is diminished by the action of light and increased by heat. It has not been ascertained whether the effect is upon the E.M.F. or the internal resistance, or both.

If a battery-current is caused to pass for a short time through a cell consisting of two silver electrodes imbedded in a mixture of sulphur and copper sulphide, the cell, after being disconnected from the battery, will generate a current of very short duration in the direction opposite to that of the battery-current, followed by a current which may be maintained for several

* Exp. Res. §§ 1911 and 2036.

hours in the *same* direction as the battery-current. A second reversal has in one case been found to occur after an interval of four hours.

The experiments, of which an account is here given, must be considered as being merely of a preliminary nature. A complete investigation of the phenomena in question would require more time than I am at present able to devote to the subject.

Addition, August 3rd.

From (4), (5), and (6) it appears that a cell consisting of copper and silver plates is incapable of generating a current unless a certain proportion of free sulphur is mixed with the sulphide. Thinking that the function of the free sulphur might be merely to form silver sulphide by direct combination with the silver, I constructed a cell as follows :—A layer of copper sulphide was spread upon a plate of copper ; a polished steel plate was laid upon the sulphide, and the whole was strongly compressed in a vice. The steel plate was then removed, and a thin layer of silver sulphide was spread upon the smooth surface of the copper sulphide. The cell was completed by pressing a silver plate upon the silver sulphide. This was found upon trial to give a current which, with an external circuit of low resistance, was many times stronger than that generated by any of the cells previously made. It seems to be exactly analogous in its action to a Daniell cell consisting of plates of zinc and copper in solutions of zinc sulphate and copper sulphate. The quantity of the copper sulphide would be gradually diminished, copper being deposited on the copper plate, while the quantity of silver sulphide would continually increase with consumption of the silver.

In conclusion, it seems probable that, by selecting such metals as experiment might prove to be better suited for the purpose than silver and copper, a cell might be constructed upon the principle of that described in the above paragraph which would be of practical and commercial value.

XXIX. *On the Law of the Electromagnet and the Law of the Dynamo.* By Silvanus P. Thompson, *D.Sc., B.A.**

It is remarkable, considering the vast amount of research that has been made, that authorities in physics are in so little agreement as to what the law of the electromagnet is. Upon the one topic that is all-essential to the electric engineer the professed physicist is either silent, or gives mathematical rules that are hopelessly wide of the observed facts, and based in most cases on no consistent theory. The expressions given by Weber† for the relation between the magnetizing force and the induced magnetism are an exception; but they are unmanageable in the extreme, and have led to no useful results. The expression given by Lamont‡ is also based on a certain consistent theory, but it appears to be entirely unknown both to physicists and to electric engineers, and is to be found only buried in the depths of Lamont's own treatise. The two mathematical formulæ which are most often given in treatises on physics as representing the relation between the magnetizing current and the induced magnetism of the electromagnet are the following:—

Lenz and Jacobi's Formula.—According to the experiments of these early investigators (1839), the magnetism of the electromagnet is simply proportional to the strength of the current and to the number of turns of wire in the coil. This may be written as

$$m = kSi,$$

where i is the strength of the current, S the number of turns in the coil, k a constant depending on shape and quality of iron, and m the strength of the pole. Joule (1839) showed this rule to be incorrect, and that as the iron became saturated, m ceased to be proportional to Si.

Müller's Formula.—Müller gave an expression of the form:—

$$m = b \tan^{-1} \frac{Si}{a},$$

* Read November 14, 1885.

† Weber, *Elektrodynamische Maasbestimmungen*, p. 572. See also Maxwell, 'Electricity and Magnetism' (2nd edition), vol. ii. p. 78.

‡ Lamont, *Magnetismus*, p. 41.

where b and a are constants, depending on the form, quantity, and quality of the core and on the form and disposition of the coils. Very similar formulæ have been used by Von Waltenhofen, Dub, Cazin, and Breguet. They are purely empirical, fail to represent the facts with accuracy, and have moreover the defect of not lending themselves conveniently to use in the equations of electromagnetic motors, dynamo-electric machines, and other kinds of electric machinery.

More recently another formula has been coming into use, and is known—for reasons which will presently appear—as Frölich's formula: it is written

$$m = \frac{i}{a+bi},$$

where a and b are again constants, b being the reciprocal of the maximum value of m. This formula, like the preceding, is a perfectly empirical expression not based in itself upon any physical theory, and it accords with the observed facts of the electromagnet better than any of the preceding formulæ. Frölich, who in 1878 was using merely Lenz and Jacobi's formula*, appears to have adopted this expression in 1880†, in consequence of the experiments which he had made with dynamo-machines. A similar formula had been used twenty-five years before by Robinson‡ to express the lifting-power of an electromagnet, and similar formulæ have been used by Oberbeck§, Fromme‖, Clausius¶, Ayrton and Perry**, and Rücker††.

Now it must be obvious that since almost the whole of electric technology deals with the electromagnet and its applications, it is of utmost importance to know what the true law of the electromagnet is. If the technologist is left to the mercy of the physicist, he has to choose between formulæ

* Frölich, *Die Lehre von der Elektricität und dem Magnetismus*, p. 237.

† Frölich, *Berl. Berichte*, Nov. 18, 1880, p. 973; also *Elektrotechnische Zeitschrift*, 1881, April, p. 139.

‡ Robinson, Trans. Roy. Irish Academy, vol. xxii. p. 1, 1855.

§ Oberbeck, *Pogg. Ann.* cxxxv. pp. 74–98, 1868.

‖ Fromme, *Pogg. Ann.* clv. p. 305, 1878.

¶ Clausius, *Wied. Ann.* xx. p. 367, 1883.

** Ayrton and Perry, Journ. Soc. Telegr. Eng. and Electr. xii. p. 317, 1883.

†† Rücker, Phil. Mag. ser. 5, xix. p. 463 (June 1885).

that are based on physical conceptions and which are utterly unmanageable for his purpose, and formulæ that are purely empirical expressions and have no rational basis.

But the study of the modern dynamo-electric machine has brought us to a point that makes some decision a necessity. The entire action of the dynamo is dependent on the magnetizing action of the electric current, and the law of the dynamo is, in the very nature of things, not to be discovered, save by the discovery of the law of the electromagnet. It is indeed extraordinary that such able physicists as Mascart and Angot, Mayer and Auerbach, Schwendler, and Herwig sought in vain for the true law to connect the electromotive force of the dynamo with its speed, the resistance of its circuit, and the constants of its construction. Hopkinson came nearest to the mark when in 1879 he described the properties of the dynamo in terms of those of a certain curve, which we now call the characteristic curve of the machine; but he did not state the law of the dynamo algebraically, and apparently he accepted as true Weber's formula for the electromagnet.

But the law of the dynamo is now known, and expressions have been deduced which are found to agree with the utmost accuracy with the observed facts. The discovery is almost entirely due to Dr. Frölich*, whose papers, published at intervals from 1880 to 1885, are in their way already classical. The results first obtained by Frölich related exclusively to that form of dynamo in which the armature-coils and the field-magnet coils are both included in the main circuit—the series-dynamo—though they have been further extended by the present writer, and by others, including Frölich himself, to other forms of machine. A very brief resumé of Frölich's research will explain his method of arriving at the law.

His theory is based upon (1) Faraday's law of induction, (2) Ohm's law, (3) a curve, called by him the current-curve, expressing certain results of experiments made on the series-wound dynamo.

Following Faraday's principle, the induced electromotive

* Frölich's chief papers are as follows:—*Berl. Berichte*, 1880, p. 978; *Elektrotechnische Zeitschrift*, vol. ii. p. 134 (1881); *ib.* vol. iii. pp. 69 and 113 (1882); *ib.* vol. iv. pp. 60, 67, and 71 (1883); *ib.* vol. vi. pp. 128, 139, 227 (1885).

force E will be proportional to the speed of the machine n and to a quantity M which Frölich calls the "effective magnetism," and which is itself proportional to the effective area of the armature-coils and to the intensity of the magnetic field. Writing this as an equation, we have

$$E = nM,$$

and, by Ohm's law,

$$E = iR,$$

which gives us for the current i,

$$i = \frac{n}{R}M,$$

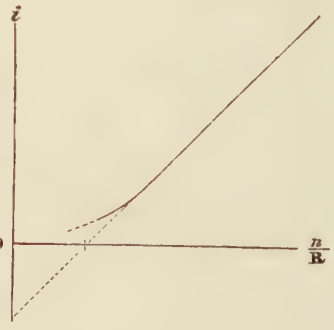
or

$$\frac{i}{M} = \frac{n}{R}.$$

Since M is itself a function of i, and not primarily of n or R, it is clear that i is itself a function of n/R, and we may write

$$\phi(i) = \frac{n}{R}.$$

This function Frölich set himself to investigate by pure experiment without making any hypothesis whatever. He determined the values of i at various speeds and with various resistances, and plotted out the results as a curve, values of i being taken as ordinates and values of n/R as abscissæ. This curve (see figure) he termed the "current-curve." It shows itself to be very nearly a straight line, which, however, does not pass through the origin.



The portion which departs from the line is difficult of observation because it is only obtained when either n is very small or R very large, that is to say when the dynamo is scarcely able to excite its magnets. Neglecting this unstable state, and dealing only with that part of the curve which relates to the machine as it is when in action, it is clear that the relation between the two variables may be expressed as

$$bi=\frac{n}{\mathrm{R}}-a,$$

where b and a are constants ; giving us as the equation of the series-dynamo :—

$$i=\frac{1}{b}\left(\frac{n}{\mathrm{R}}-a\right).$$

Substituting for n/R its equivalent i/M, Frölich then obtained from the law of the dynamo the following expression for the "effective magnetism":—

$$\mathrm{M}=\frac{i}{a+bi},$$

an expression which he himself appears, in his original paper, to have considered as a mere interpolation-formula. It is, however, this expression which is generally referred to as Frölich's formula for the electromagnet.

It was in 1883 that the author of this paper, in working at the problem of the dynamo, realized the significance of this result of Frölich's. He had been using as the fundamental law

$$\mathrm{E}=4n\mathrm{AH}$$

to express the average value of the electromotive force developed by rotating in a uniform magnetic field of intensity H, an armature whose total effective area was A ; and requiring a formula to connect H with the exciting current, he was considering the various equations of Müller, Weber, and others when he became aware that Frölich's expression furnished what was needed ; and, finding it necessary for the case of compound-wound machines to express the excitation in terms of ampère-turns rather than of ampères, he rewrote the formula in the following way :—

$$\mathrm{H}=\mathrm{GS}i\frac{k}{1+\sigma\mathrm{S}i},$$

where, as before, Si is the number of ampère-turns, G a geometrical coefficient, k the coefficient of magnetic permeability in the initial stage, and σ a small saturation-coefficient dependent on the quantity and form as well as on the quality of the iron of the core.

Writing similarly Zi_s for the ampère-turns of a shunt-winding, the expressions for H became

$$H = GZi_s \frac{k}{1+\sigma Zi_s}$$

for shunt-wound machines, and

$$H = G(Si \pm Zi_s) \frac{k}{1+\sigma(Si \pm Zi_s)}$$

for compound-wound machines.

Inserting these values in the fundamental equation at once gave the following expressions :—

Series dynamo.

$$E = \frac{1}{\sigma}\left\{4nAGk - \frac{\Sigma R}{S}\right\}$$

$$i = \frac{1}{\sigma}\left\{\frac{4nAGk}{\Sigma R} - \frac{1}{S}\right\}$$

Shunt dynamo.

$$E = \frac{1}{\sigma}\left\{4nAGk - \frac{r_a r_s + R r_s + R r_a}{ZR}\right\}$$

Compound dynamo (at critical speed for regulation).

$$e = \frac{1}{\sigma}\left\{\frac{r_s(r_a+r_m)}{S(r_s+r_a)} - \frac{r_s}{Z}\right\} \text{(short shunt)},$$

or

$$e = \frac{1}{\sigma}\left\{\frac{r_s(r_a+r_m)}{S(r_s+r_a+r_m)} - \frac{r_s}{Z+S}\right\} \text{(long shunt)},$$

where r_a, r_m, r_s, and R are the resistances of armature, main-circuit coil, shunt-coil, and external circuit respectively, and e the difference of potentials at the terminals of the machine. These results were announced by the author at the Montreal meeting of the British Association, and were published by him in his volume on Dynamo-Electric Machinery, as was also the important proposition that the factor $\frac{1}{\sigma}$ (which may be called *the determinant* of the working power of the dynamo, since it appears in every one of the equations, both of current and electromotive force) is that number of ampère-turns which will reduce the effective permeability of the

electromagnet, or rather of the magnetic circuit, to half its initial value. It will be seen that all these synthetical results were new, except that for the current of the series dynamo, which is identical with the expression that had been deduced by Dr. Frölich from the current-curve which represented his experimental results.

Since the publication of the author's treatise, Frölich has further elaborated the equations of the compound-wound machines; and a more generalized form has been deduced by Prof. Rücker in the masterly paper which he communicated to the Physical Society in March last, in which, however, he attributes to Frölich (I know not on what grounds) the general equation for the electromotive force of all kinds of dynamos. Frölich has also verified the adequacy of these formulæ by comparing results calculated from them with actual experiments on compound-wound machines. One example will suffice.

n.	R.	*e* obs.	*e* calc.	*i* obs.	*i* calc.
850	0·841	127	127	151·0	151·0
853	1·22	133	133	109·0	109·0
855	2·34	139	140	59·9	59·4
850	4·52	138	139	30·3	30·5
850	9·25	136	138	14·7	14·7
850	22·1	137	136	6·2	6·2
845	222·0	133	134	0·6	0·6

Such a series of results proves the law of the dynamo to be fully established. But if the law of the dynamo, so deduced from Frölich's empirical expression for the electromagnet, be true, *then*, to an equal degree of precision, *must the law of the electromagnet be itself true.*

But Frölich's expression is not the expression of any physical law: it is a mere interpolation-formula, destitute of rational significance. What then is the true law of induced magnetism? And how comes it that an expression devoid of physical significance so nearly expresses the physical law?

That question the author believes to have been already

solved by a forgotten investigation of Lamont, published almost without note or comment in his *Handbuch von dem Magnetismus* in 1867. On p. 41 of that work he gives the following equation:—

$$m = \frac{a\mathrm{M}x}{\mathrm{M} + ax},$$

as a convenient first aproximation to another equation based on a new theory of induced magnetism. In the above equation x stands for the magnetizing force, m the induced magnetism, and M the maximum value of the latter. Writing Gk for a, Si for x, and σ for $\frac{a}{\mathrm{M}}$, we at once get

$$m = \frac{\mathrm{G}k\mathrm{S}i}{1 + \sigma\mathrm{S}i},$$

which is the author's way of writing Frölich's equation.

The physical theory which led Lamont up to this result is of great interest, the more so since it has been revived during the current year in a more elaborate form by Bosanquet*, who has apparently been led thereto by an independent line of thought, and who has gone much further than Lamont in bringing in the additional consideration of the forces resisting the orientation of the molecules. Lamont's theory is, briefly, that the permeability of the iron diminishes as the permeation increases, being at every stage of the magnetization proportional to the deficit of saturation. He assumes that for every bar there is a certain maximum of magnetization, to which it could only attain under the influence of an infinitely great magnetizing force; and that the permeability of the bar is at every stage of the magnetization proportional to the difference between the actual magnetization and the possible magnetization. This is, in other words, as if in every bar there were room for only a certain limited number of magnetic lines of force, and that, when any lesser number have been induced in it, the susceptibility of the bar to the reception of additional lines is proportional to the room yet left for them in the bar. Lamont's theory is expressed as follows: Let the magnetism present at any stage be called m, and let the maximum magnetism be called M. Then the

* Bosanquet, Phil. Mag. ser. 5, xix. p. 85 (Feb. 1885).

amount which the bar can still take up is $M-m$; and it is to this that the permeability dm/dx is proportional. Here x may be understood, as itself proportional to the number of ampère-turns of the exciting current; and we may write

$$\frac{dm}{dx}=k\,(M-m),$$

where k is a constant depending on the units employed, and on the initial value of the permeability of the magnetic circuit when $m=0$. Integrating we obtain

$$M-m=A\epsilon^{-kx},$$

where A is a constant of integration. But when $x=0$, $m=0$ also; whence $A=M$, giving

$$m=M(1-\epsilon^{-kx}),$$

which is Lamont's formula.

Lamont also develops this expression in ascending powers of kx,

$$m=Mkx\left(1-\frac{kx}{1.2}+\frac{k^2x^2}{1.2.3}-\ .\ .\ .\ .\ .\right),$$

and he then makes the remark that it may be approximately represented by the simpler, but empirical expression

$$m=\frac{aMx}{M+ax},$$

which is identical in form with the expression known as Frölich's. Now, writing k for a/M,

$$m=\frac{Mkx}{1+kx},$$

and expanding this in ascending powers, we get

$$m=Mkx(1-kx+k^2x^2-\ .\ .\ .\ .\ .).$$

Neglecting the fourth and higher terms, it will be seen that the formulæ are very nearly equal for all small values of kx, and are identical for the value

$$kx=\frac{3}{5},$$

or when the actual magnetism has about ·456 of the value it would have under an infinite magnetizing force. For larger values of the magnetizing force the values of m calculated by

the empirical formula are slightly greater than those calculated by the exponential formula.

None of these formulæ render any account of a phenomenon noticed in many cases where an electromagnet is magnetized with gradually ascending magnetic forces, namely, a concavity in the *early* part of the curve of magnetization, the permeability apparently becoming greater after a certain degree of magnetization has been attained. The researches of Chwolson and of Siemens seem to show that this apparent increase in the permeability (which is not observed with descending magnetic forces) is due to non-homogeneity, and to the resistance of some of the molecules to magnetization. Bosanquet's theory may be taken in connection with the recent observations of Rowland, Warburg, Ewing, and Hopkinson on this subject, which, however, is of no great importance here since the apparent maximum of permeability is attained in most cases at a much lower degree of magnetization than that at which the dynamo is actually worked.

The very close agreement between the observed values of current and potential of the dynamo and those calculated by the equations based upon Frölich's formula, proves that nowhere within the working range do the actual values of the magnetization differ sensibly from those calculated by the Frölich formula. That is to say this formula may be taken as *something more than a first approximation* to the true law of the electromagnet for all degrees of saturation of the magnets employed in practice, and may be taken as a first approximation for all degrees of saturation beyond that usually attained.

It is eminently desirable, in the face of these facts, that the imperfect formula of Lenz and Jacobi, as well as that of Müller, should henceforth disappear from the text-books of physics, and that instead thereof the true law of the electromagnet should be stated, either in the exponential form or in the simpler form that is found to be equally true within the range of saturation attained in practice.

XXX. *A Note on the Electromotive Force of certain Tin Cells.*

By E. F. Herroun, *of King's College, London.**

Having failed to find any published account giving the electromotive force of cells consisting of *tin* in solutions of any of its salts, opposed to other metals in solutions of their corresponding salts, I thought that some measurements recently made by myself might be of interest to this Society as tending to complete the series of double-fluid cells already investigated and recorded by others.

The cells examined consisted of an outer containing-vessel of glass or glazed earthenware, in which was placed one of the metals in a solution of one of its salts, and an inner porous pot, 8 centim. high, containing the other metal in a solution of its corresponding salt. It will be seen that they were, therefore, similar in construction to the ordinary laboratory form of Daniell cell, and the standard employed in these experiments was a Daniell cell made up in exactly the same manner.

The solutions in the Daniell cell were copper and zinc sulphates of equal molecular strength as recommended by Dr. C. Alder Wright, the proportion being 1·8 $M''SO_4$, 100 H_2O. The cell made up in this way was assumed to have an electromotive force of 1·08 volt, and it probably differed but little from this value.

Method of Measurement.

The measurements of the E.M.F. of the cells were made by the zero method of Latimer Clark, in which no polarization is possible when the adjustment is effected. The standard Daniell cell was used before and after each experimental cell in order to avoid error due to variation in the potential of the wire, resulting from changes in the E.M.F. or internal resistance of the third or polarizing battery, which consisted either of two Daniell cells or a Fuller's bichromate. This precaution, however, was usually found to be superfluous, as the circuit of the third battery was only closed when an observation was

* Read November 14, 1885.

being made, and its E.M.F. and internal resistance varied only within very small limits.

In order to ascertain if the E.M.F.'s measured were really due to the essential chemical changes occurring within the cell, it was usual, after making an observation with an experimental cell when first put up, to allow the cell to work for a short time on a circuit of a few ohms, so as to cause the surfaces of the metals to become dissolved and deposited upon respectively. The circuit was then opened, and after a short rest the E.M.F. was again determined. If this differed by little or nothing from the previous measurements it was considered to be the true value for that cell ; if, on the contrary, there was an appreciable difference further measurements were made. By operating in this way it was found that copper plates (particularly electro-deposited metal) which had been exposed to the air, and had acquired a superficial film of suboxide, usually gave, when first put up, an E.M.F. somewhat in excess of their true value. After allowing the cell to send a current for a few minutes, the E.M.F. fell to the normal value, and thereafter remained sensibly constant. (*Vide* also Dr. Fleming, "Use of Daniell's Cell," Proc. Phys. Soc. vol. vii. p. 168.)

As such initial fictitious electromotive forces would affect to their full extent the results obtained by a method like that of Poggendorf or Latimer Clark, or in fact any method in which difference of potential and not the current is measured, it was thought well to control such measurements with one or two cells by using Wheatstone's method of adding a resistance to bring down the deflection of a galvanometer-needle through a given number of degrees. The results obtained by this means agreed remarkably with those found by the previously mentioned process.

In the calculations relating to the heat of formation of some of the salts, it has been taken as a mean value that the electrochemical equivalent of hydrogen is ·0001038*, and that the Joule equivalent is $4{\cdot}16 \times 10^7$ ergs. From these numbers the factor which, multiplied into the heat, expressed in gramme-degrees, evolved by the displacement of the

* "On the Determination of Chemical Affinity in terms of Electromotive Force," by Messrs. Wright and Thompson, Phil. Mag. vol. xix. p. 7.

gramme equivalent of one metal by that of another, will give the electromotive force in volts, is found to be $4{\cdot}318 \times 10^{-5}$ or $4{\cdot}32 \times 10^{-5}$ nearly.

The electromotive forces of the following combinations have been measured :—tin and copper, zinc and tin, cadmium and tin, in solutions of their sulphates and chlorides; and zinc and tin in solutions of their iodides.

The stannous sulphate employed was obtained by the action of sulphuric acid with the aid of heat upon finely granulated tin, in presence of platinum or copper to aid the action. The solution of stannous sulphate contained some free sulphuric acid ; and in fact it appears to be impossible to prepare a neutral solution suitable for these experiments, as on dilution to the required strength it deposits a basic salt, the solution becoming acid. Moreover a nearly neutral solution of stannous sulphate when kept is easily acted on by atmospheric oxygen, becoming cloudy and finally depositing what is probably either a mixture of basic stannous sulphate and stannic hydrate, or a basic stannic sulphate (?). With free sulphuric acid present in the proportion of from 1 to 2 per cent., the solution was satisfactory, and the effect of the free acid is considered below. For similar reasons it was found necessary to operate with solutions of stannous chloride containing a small amount of free hydrochloric acid, and with stannous iodide containing free hydriodic acid.

The metallic tin employed as plates in these cells was chemically pure, it having been obtained by electrolysis of a pure solution of the sulphate. The electro-deposited metal was carefully fused and rolled into strips.

The copper plates employed had in all cases been well covered with the electro-deposited metal. The zinc used was a good specimen of commercially "pure" metal, and was amalgamated. The cadmium was fairly pure, and was covered with the electro-deposited metal.

Electromotive Force of Cells consisting of the Metals immersed in Solutions of their Sulphates.

I. *Zinc-Tin Sulphate Cell.*

A cell in which zinc in $\cdot 5ZnSO_4$, 100 H_2O is opposed to tin

in ·5$SnSO_4$, 100 H_2O, *both* solutions containing about 1·5 per cent. H_2SO_4, was found to have an E.M.F. of from ·519 to ·535 volt, the variation being due to two causes: (1) slight differences in the percentage of free H_2SO_4, and (2) differences in the condition of the surface of the tin plate.

It is thus difficult to assign an exact value except between such limits; but with solutions of the above strength, and with a tin plate covered with freshly deposited tin, the value of ·525 volt may be taken as a tolerably accurate average E.M.F. for this kind of cell.

It would, of course, be unphilosophical to assume that an *equality* in the amount of free acid present in each solution has the same effect as operating with neutral solutions; but being unable to employ a neutral solution of stannous sulphate, the addition of an amount of sulphuric acid to the zinc solution equal to that present in the $SnSO_4$ solution was considered to be the fairest way of meeting the difficulty.

Increasing the amount of free sulphuric acid in the tin solution only has naturally the effect of lowering the E.M.F., so that with 5 per cent. H_2SO_4 the E.M.F. is not higher than ·49 volt.

Substituting a 10 per cent. solution of H_2SO_4 for the zinc sulphate, leaving the stannous sulphate unchanged, raises the E.M.F. to ·596 volt.

The effect of varying the strength of both solutions (keeping them of equal molecular strength) is the same in kind as that which occurs with the zinc-tin chloride cell, but is less marked. (See Zinc-Tin Chloride Cell.)

The heat of formation of stannous sulphate being unknown, it is impossible to compare the value ·525 volt with deductions from thermal data.

II. *Tin-Copper Sulphate Cell.*

A cell in which tin in ·5$SnSO_4$, 100 H_2O is opposed to copper in ·5 $CuSO_4$, 100 H_2O, containing from 1 to 2 per cent. H_2SO_4 in both solutions, was found to have an E.M.F. varying from ·56 to ·572 volt, according to the condition of the surfaces of the plates and the acidity of the solutions. The effect of increasing the proportion of free H_2SO_4 in the tin solution is to raise the E.M.F. of the cell, and it is, of

course, opposite to the effect observed in a cell in which zinc displaces tin. The increase of free sulphuric acid equally in both tin and copper solutions also tends (at least through a certain range up to 3 or 4 per cent.) to slightly raise the E.M.F.; and from this I conclude that with neutral solutions of the above strength, the value would not exceed, and would probably fall slightly below, ·56 volt. Preference would therefore be given to the minimum value ·56 instead of the mean value ·566.

Alteration of the strength of the solutions is complicated by the question whether such alteration involves a change in the percentage of contained acid or only of the salts employed. Increase of strength of the solutions as regards the salts only, is attended with a slight depreciation of the E.M.F.; but if the addition of more salt causes an increase in the acidity of the tin solution, the effect is to heighten the E.M.F., so that these opposite effects may equal each other, or one may exceed the other, according to their relative magnitude.

Cells were set up in which tin in a solution of 20 per cent. H_2SO_4 (sp. gr. 1·141) was opposed respectively to tin in ·5$SnSO_4$, 100 H_2O, containing 1½ per cent. H_2SO_4, and to ·5$CuSO_4$, 100 H_2O, and the following values obtained :—

Tin-tin sulphate cell.	·058–·062 volt.
Tin sulphuric-acid-copper sulphate cell.	·59 volt.

(It will be seen that the last-mentioned cell does not exceed the tin-copper sulphate cell (·566) by an amount equal to the tin-tin cell, the lower E.M.F. being probably due to an adverse E.M.F. resulting from diffusion into the comparatively strong sulphuric-acid solution.)

If the mean values assigned to the zinc-tin cell and the tin-copper cell are added together, they are found to exceed the zinc-copper cell, which they should theoretically equal

$$(\cdot525+\cdot566=1\cdot091-1\cdot08=\cdot011 \text{ excess}).$$

This result confirms the opinion that for neutral solutions the value ·566 is probably too high for the tin-copper cell, and justifies the minimum value ·56 being taken as correct. This still gives a slight excess $1\cdot085-1\cdot08=0\cdot005$, and in practice it is found that a zinc-tin cell and a tin-copper cell

united in series actually give an E.M.F. slightly in excess of the zinc-copper cell. This difference may be due either to the different conditions of the surfaces of the two tin plates or to some less obvious cause; but as it has been found to exist on several occasions in different cells, it can hardly be a mere coincidence.

III. *Cadmium-Tin Sulphate Cell.*

A cell set up with cadmium in $\cdot 5CdSO_4$, 100 H_2O, opposed to tin in $\cdot 5SnSO_4$, 100 H_2O, both containing about 1·5 per cent. H_2SO_4, was found to have an E.M.F. of ·189 volt.

In order to compare the summation with the zinc-tin cell, a determination of the zinc-cadmium sulphate cell was made (using $\cdot 5M''SO_4$, $100H_2O$), and it was found to have an E.M.F. of ·335 volt. (The value given by Messrs. Wright and Thompson for the zinc-cadmium cell where $m = \cdot 5$ and the solutions are *neutral* is ·361, which, correcting for difference of standard, gives $\frac{1\cdot 08}{1\cdot 114} \times \cdot 361 = \cdot 349$ or somewhat higher.)

Adding together the values obtained we get $\cdot 189 + \cdot 335 = \cdot 524$ volt for the zinc-tin cell, a result which shows a striking agreement with the average value previously given, viz. ·525.

The average E.M.F. of the cadmium-copper cell was also determined and found to be ·744, Messrs. Wright and Thompson finding ·753 for the same cell, which gives $\left(\frac{\cdot 753 \times 1\cdot 08}{1\cdot 114} = \right)$ ·73 volt; or, in other words, my value for the cadmium-copper is as much above, as my zinc-cadmium is below, their estimate, both values adding up correctly,

$$\cdot 73 + \cdot 349 = 1\cdot 079,$$

and

$$\cdot 744 + \cdot 335 = 1\cdot 079.$$

This discrepancy is no doubt due to the fact that the solutions employed by me contained about 1·5 per cent. of free H_2SO_4 in order to keep the conditions uniform with the tin cells.

Subtracting the E.M.F. of cadmium-tin from that of cadmium-copper, we get the E.M.F. of tin-copper, thus :—

$$\cdot 744 - \cdot 189 = \cdot 555 \text{ volt};$$

which accords fairly well with ·56, but is, as would be anticipated, somewhat less.

Cells the algebraic sum of whose E.M.F.'s should equal 1·08, the standard Daniell.

Zinc-tin-sulphate	+	tin-copper-sulphate.				
·525	+	·56			=	1·085.
Zinc-cadmium	+	cadmium-tin	+	tin-copper.		
·335	+	·189	+	·56	=	1·084.
Zinc-tin	+	cadmium-copper	−	cadmium-tin.		
·525	+	·744	−	·189	=	1·08.

Electromotive Force of Cells consisting of the Metals in Solutions of their Chlorides of Equal Molecular Strength.

I. *Zinc-Tin Chloride Cell.*

A cell set up with amalgamated zinc opposed to tin in solutions of their chlorides of the strength ·5 $M''Cl_2$, 100 H_2O, containing 1 per cent. HCl, gave an E.M.F. of ·544 to ·553 volt, mean = ·549 volt.

Diluting the solutions equally with distilled water causes a rise of E.M.F. up to about ·607 volt, but dilution with 1 per cent. hydrochloric acid causes a slight diminution of E.M.F. This and other facts render it probable that with this cell, and to a less extent with the zinc-tin sulphate cell, increase of solution-strength without increase in the percentage of contained acid is associated with a rise of E.M.F.; but that with a corresponding increase of acidity, with a fall of E.M.F. Thus it is that dilution with distilled water raises the E.M.F. not because the metallic salt is diluted, but because the free acid is more attenuated.

Mutatis mutandis the same remarks apply to tin-copper or other cells in which tin is the metal attacked (*i. e.* with constant acidity and increasing strength of solution the E.M.F. falls, but with increase of *both* the E.M.F. rises, or does not fall).

The heat of formation of zinc chloride and of stannous chloride being known, the theoretical E.M.F. is directly calculable.

$[Zn, Cl_2, aq.] = 112840.$ $[Sn, Cl_2, aq.] = 81140.$

112840 − 81140 = 31700; half of which multiplied into $4{\cdot}32 \times 10^{-5}$ gives ·684 volt as the E.M.F. of zinc displacing tin from solution of its chloride, and the difference between the observed and calculated value is ·684 − ·549 = ·135 volt. Or to employ the terminology of Messrs. Wright and Thompson, −·135 is the "*Thermovoltaic constant*" for tin in solution of its chloride for certain strengths of solution (near ·5 MCl_2, 100 H_2O).

II. *Cadmium-Tin Chloride Cell.*

A cell set up with electro-cadmium opposed to tin in solutions of their chlorides, ·5 $M''Cl_2$, 100 H_2O, was found to give an E.M.F. of ·247 to ·262 volt and an average value of ·249 volt. The solutions used contained the same amount of free hydrochloric acid, namely about 1 per cent.

The heats of formation being [Cd, Cl_2, aq.] = 96250 and [Sn, Cl_2, aq.] = 81140 and their difference 15110 or 7555 per equivalent, the theoretical E.M.F. of cadmium displacing tin from dilute solutions of its chloride is ·326 volt, and the difference between this and the average observed value is ·077.

The E.M.F. of a zinc-cadmium chloride cell was also determined and found to be between ·291 and ·308 volt, giving an average of ·301 volt. This is below the value given by Messrs. Wright and Thompson, owing to the solutions used by me containing 1 per cent. free HCl, they giving for this cell ·329 volt, which becomes $\left(\frac{\cdot 329 \times 1{\cdot}08}{1{\cdot}114} = \right)$ ·318 volt, allowing for difference of standard.

In reference to a point to which these authors* drew attention a short time ago, namely, that, although the heat evolved by displacing cadmium by zinc in solutions ·25 $CdCl_2$, 100 H_2O is greater than that evolved by the same substitution in the case of the sulphates, yet the E.M.F. of the zinc-cadmium *chloride* cell is materially less than that of the zinc-cadmium *sulphate* cell, my measurements afford direct confirmation, as the value given here for the average E.M.F. of the zinc-cadmium sulphate cell is ·335 volt and that for the zinc-cadmium chloride cell only ·301 volt, or a difference of ·034 volt, substantially the same as the difference given by them, viz. ·035.

* Phil. Mag. vol. xvii. p. 378 (May 1884).

III. *Electromotive Force of Zinc-Tin Iodide Cell.*

A cell set up with amalgamated zinc in ·25 ZnI_2, 100 H_2O opposed to tin in ·25 SnI_2, 100 H_2O, both solutions acidified with 1·5 per cent. of HI, was found to have an E.M.F. of from ·47 to ·49, averaging ·485 volt.

There being no thermal data at my disposal no comparisons can be drawn.

XXXI. *Note on the Verification of Thermometers at the Freezing-Point of Mercury. By* G. W. Whipple, *B.Sc., F.R.A.S., Superintendent of the Kew Observatory**.

It is not many years since the operation of freezing mercury was considered a scientific feat, and a party of gentlemen were invited to the Kew Observatory to assist at the performance of the operation by the late Mr. R. Adams. That gentleman produced the solidified mercury by the action of ether upon carbonic acid in the solid form, and with his assistance considerable quantities of mercury were frozen, and a number of observations were made with carefully constructed thermometers, which gave the value for the melting-point of mercury which has since been accepted as the fixed point. Prof. Balfour Stewart communicated an account of these experiments to the Royal Society, in whose publications they will be found (Phil. Trans. 1863, pp. 425–435).

Since that date the determination of the low-temperature fixed point of thermometers has become a regular operation in the verification department of the Kew Observatory, and some eight or nine dozen instruments are compared in melting mercury annually, most of them being destined for use in Canada and other cold countries.

As the mode of making this comparison is not described in any readily accessible publication, we have frequently been requested to give an account of it, and have accordingly drafted this paper with a view of putting it on record.

The carbonic acid in its liquid form is purchased generally of Orchard, of High Street, Kensington, in the well-known

* Read November 14, 1885.

heavy iron bottles so familiar in lecture-rooms as recipients of compressed oxygen. A convenient quantity for the prosecution of an experiment is 200 gallons of gas, which weighs about 3 lb. The quantity of mercury operated upon is from 17 to 20 lb. By means of a common salt and ice freezing-mixture, the vessels, thermometers, mercury, and ether are all cooled down to about $-10°$ C. beforehand.

The collecting-box being applied to the nozzle of the gas-receiver, the valve is unscrewed and gas allowed to escape through the apertures in the handles of the box for about a minute, during which time about 550 grains of solid carbonic gas are deposited in the form of snow.

The box being opened, the snow is removed with an ivory paper-knife and laid on the surface of the mercury previously placed in a wooden box well jacketed with felt. 2 oz. of ether is then poured over it, and the pasty mass forced under the surface of the mercury by means of a special wooden stirrer. The liberated gas and ether-vapour escaping through the mercury rapidly cool its temperature down, and a thin film of solid mercury forms on the top of the mass, which is immediately broken up by the stirrer. About six charges of acid snow and six fluid ounces of ether are usually necessary to convert the whole of the mercury into the finely granular mass of a pasty consistence.

It is considered inadvisable to freeze the mercury into a solid state, as then it becomes necessary to take it out of the box, and cut it up with shears into small lumps before the thermometers can be inserted into it, an operation which entails great loss of time and frozen mercury.

When the mass is in the semi-solidified condition the thermometers are plunged into it and allowed to fall to the requisite temperature—a well-verified standard being always placed in the bath at the same time and carefully watched during the progress of the comparison.

As soon as this standard is seen to begin to rise above the lowest point, the thermometers are taken out, and an additional quantity of snow and ether forced into the mercury until its temperature is again reduced, and the *feel* of the thermometer shows it has regained the proper consistency.

When the operation is properly and carefully performed, it

is found that the work of comparison may be spread over from half an hour to forty minutes, a period which permits of more than 100 mercurial thermometers being read with the necessary exactitude. If the thermometers are ordinary alcohol or Rutherford minimums with large bulbs, only from thirty to forty can be compared at a time.

The operation might be continued for an indefinite period were a sufficient supply of liquefied carbonic acid available, but at Kew we usually find the 200-gallon bottle, which, as already referred to, contains about 3 lb. weight of gas at sufficient pressure to afford snow, adequate for our purpose.

The cost of the materials is as follows :—gas 19*s.* 2*d*; ether 1*s.* 2*d.*

We do not wish to claim any credit for the process above described, but only to bring before the Members of the Society a description of our *modus operandi* as a matter which may perhaps be of a little interest to some of them.

XXXII. *Note on the Calibration of Galvanometers by a Constant Current.* *By* T. MATHER, *Assistant in the Physical Department, City and Guilds of London Central Institution**.

THE principle of the method to be described consists in measuring the turning moment exerted on the magnetic needle of the galvanometer by a constant current, when the needle and coils are in different relative positions.

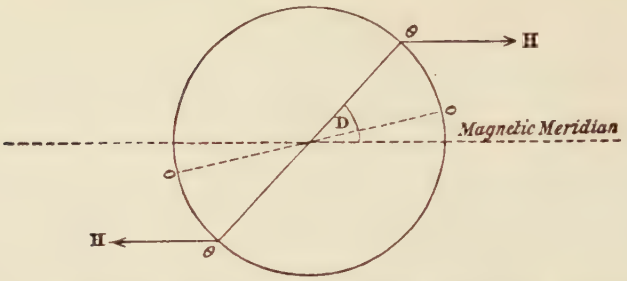
Suppose the galvanometer is placed in any position in a uniform field, and that a suitable current be passed through it, which current we will take as our *unit* current. The needle will take up some position of equilibrium, say, at an angle D to the magnetic meridian, and at angle θ to the zero of the scale (fig. 1).

When in this position, the controlling couple is :: to sin D ; and since this is balanced by the deflecting couple due to unit current, we see that this couple due to unit current, when the needle points to θ is also :: to sin D (fig. 1).

But in the ordinary use of the instrument the line of zeros

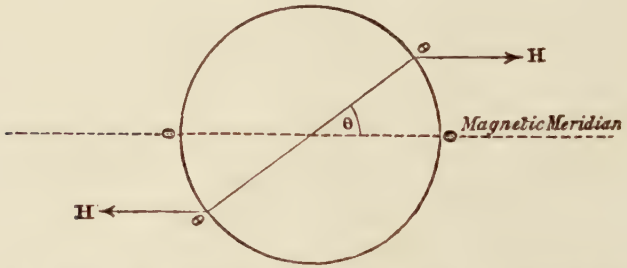
* Read November 28, 1885.

Fig. 1.



is in the magnetic meridian, and when the needle is deflected through angle θ the controlling couple is :: to $\sin\theta$. Hence the current required to produce this deflection θ when the galvanometer is used in the ordinary way, is $\frac{\sin\theta}{\sin D}$. For the currents must be proportional to the controlling couples, since the relative positions of the needle and coils are the same (fig. 2).

Fig. 2.



This being true for any deflection, the calibration-curve is got by plotting θ and $\frac{\sin\theta}{\sin D}$.

Of course the same strength of current must be used in determining the different values of D.

To measure the angle D corresponding to any θ, we have merely to break the circuit and observe the angle through which the needle moves before coming to rest in the magnetic meridian. θ is read off directly on the scale of the galvanometer when the current is passing.

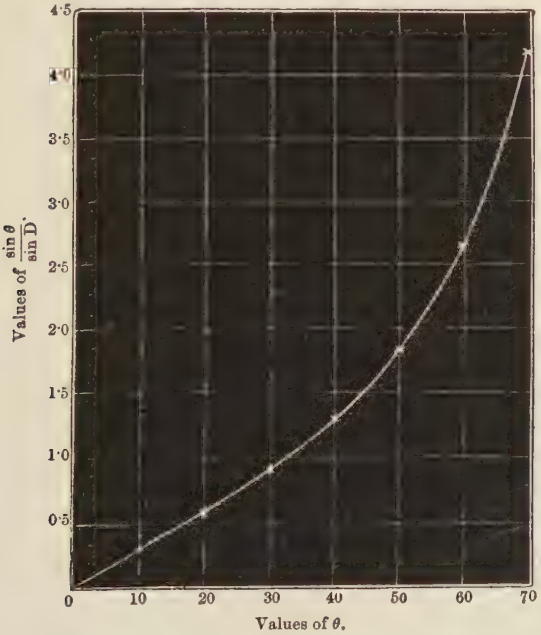
Below are appended the results of a calibration obtained by

this method, for which I have to thank Mr. Kilgour, a student of the Central Institution, who made the observations :—

θ.	D.	$\frac{\sin\theta}{\sin D}$.
10	34·8	·304
20	35·3	·592
30	32·2	·938
40	29·2	1·318
50	25·0	1·813
60	19·2	2·633
70	12·9	4·209

If θ and $\frac{\sin\theta}{\sin D}$ be plotted, we get the calibration-curve for that particular galvanometer (fig. 3).

Fig. 3.



In the foregoing Table the values of θ are integral numbers of degrees; but this is not at all necessary, for no fine adjustment, such as is required with sine-galvanometers, need be attempted. All that is needed is that the position of the needle, as indicated on the scale of the galvanometer, should

be accurately noted before breaking the circuit to observe the corresponding D.

The method described above may be utilized as a ready means of testing whether a galvanometer, whose law is *supposed to be known*, does *actually* conform thereto. For if the law of the galvanometer is $C=f(\theta)$, then, on plotting $f(\theta)$ and $\frac{\sin\theta}{\sin D}$, a straight line should result.

In the case of tangent-galvanometers, where $f(\theta)=k\tan\theta$ (k being a constant), we have to plot $\frac{\sin\theta}{\sin D}$ and $\tan\theta$ (*i. e.* $\frac{\sin\theta}{\cos\theta}$). Thus $\sin\theta$ is common to both variables ; and hence, if we plot sin D and $\cos\theta$, we should get a straight line if the galvanometer be really a tangent one.

About the law of sine-galvanometers there can be no question if the controlling field be uniform ; and herein lies one great advantage of sine-galvanometers.

In "proportional" galvanometers, where the deflection is supposed to be proportional to the current, the curve obtained by plotting θ and $\frac{\sin\theta}{\sin D}$ should be a straight line ; or

$$\frac{\sin\theta}{\theta\sin D}=\text{a constant},$$

if calculation be preferred.

It may be pointed out that, by choosing our unit current such that the maximum value of D is nearly 90°, the method admits of great sensibility.

The quantity of apparatus required to perform the calibration is reduced to a minimum, nothing but the galvanometer and a suitable current-generator being needed.

As described above, the angle D is measured by breaking the circuit and allowing the needle to come to rest in the magnetic meridian. This requires some time, if the needle is not dead-beat, and the observations could be greatly facilitated by the use of a fixed divided circle or pointer, by which D could be read off simultaneously with θ without breaking the circuit. If so arranged, the whole operation could be completed in a few minutes ; but as the method is chiefly

characterized by the fewness and simplicity of the apparatus required, it was thought undesirable to mar that simplicity by the addition of something not essential.

In conclusion, I desire to thank Professor Ayrton, F.R.S., for his kindness in suggesting the simple way of describing the method adopted in this note, and also for the statement of a problem which led to the results herein stated.

XXXIII. *Some Thermodynamical Relations.*—Part I. *By* William Ramsay, *Ph.D., and* Sydney Young, *D.Sc.**

The relations to be considered in the following pages are, we believe, well founded; but we hope to confirm them by more exact experiments than have as yet been made.

It appears advisable here to state the share which each of the authors has had in this work. For our purpose, the equation

$$\frac{L}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{J}$$

will be employed. The relations of the first term of this equation were the subject of a communication to the Chemical Section of the Philosophical Society of Glasgow in the year 1877 by Dr. Ramsay, which, however, he considered too incomplete to be published; while the application of the last term of the equation to the vapour-pressures of substances, which formed the subject of a joint research by both authors, was independently discovered by Dr. Young. It is also right here to mention that some of the relations discovered by Dr. Ramsay have been pointed out subsequently and independently by Trouton (Phil. Mag. 1884, vol. xviii. p. 54).

The term $\frac{L}{s_1 - s_2}$ (where L represents the heat of vaporization of a liquid or solid substance at its boiling- or volatilizing-point, and where s_2 is the volume of the liquid or solid at these temperatures, and s_1 that of the gas into which one or other is converted), if stated in words, denotes the heat

* Read November 28, 1885.

expressed in units required to produce unit increase of volume of substance at the temperature of ebullition of the liquid or volatilization of the solid.

Two laws have been discovered, representing certain relations between different liquids.

The first law may be stated thus:—*The amount of heat required to produce unit increase of volume in the passage from the liquid to the gaseous state at the boiling-point under normal pressure is approximately constant for all bodies;* or $\frac{L}{s_1 - s_2} = C$.

The data on which this law is based are imperfect. The heats of vaporization of but few bodies have been determined with accuracy; and we have recently shown in a communication to the Royal Society that the results obtained by Favre and Silbermann, by Regnault, and by Andrews for ethyl alcohol are only approximations to truth. It is probable that the data given for other substances less easy to obtain in a pure state are still less to be relied on. We have also shown in that paper that the density of saturated vapour of alcohol is normal, or nearly so, only at temperatures below 50°. Nevertheless, in calculating by means of the above-mentioned formula the constant for alcohol, it has been assumed that the saturated vapour of alcohol possesses normal density at 78°·4; and from want of knowledge it has similarly been assumed that at their boiling-points the saturated vapours of the other liquids to be considered have also normal density. It must therefore be acknowledged that considerable doubt rests on both the expressions in the term $\frac{L}{s_1 - s_2}$, and this doubt prevented the previous publication of these relations; but it will be seen in the sequel that the much more trustworthy determinations of vapour-pressures amply confirm the law which has been stated.

The following table exhibits these relations at normal pressure:—

Substance.	$s_1 - s_2$.	L.	$\frac{L}{s_1 - s_2}$.
Water	1695	537	0·3166
Methyl alcohol	865·9	263·7	0·3046
Ethyl alcohol	624·7	202·4	0·3240
Ethyl oxide	322·3*	90·5	0·2808
Methyl formate	417·1	117·1	0·2808
Methyl acetate	362·5	110·2	0·3040
Ethyl acetate	323·0	92·7	0·2870
Ethyl oxalate	256·2	72·7	0·2835
Amyl alcohol (?)	375·5	121·4	0·3238
Acetic acid	318·0†	101·9	0·3204
Benzene	370·6	92·3	0·2491
Bromine	169·2	45·6	0·2695
Phosphorus chloride	209·5	51·4	0·2453
Carbon disulphide	343·3	86·7	0·2526
Methyl iodide	181·5	46·1	0·2539
Ethyl iodide	180·5	46·9	0·2598
Ethyl chloride	361·9	100·1	0·2767
Carbon tetrachloride	186·3	46·9	0·2515
Chloroform	229·1	61·2	0·2673
Mercury	258·0	77·5	0·3004
Nitric peroxide	309·6‡	93·48	0·3019

It will be seen that these numbers vary between 0·2453 for phosphorus chloride and 0·3240 for ethyl alcohol. The alcohols, water, and acetic acid give all nearly 0·32; and benzene, and bodies containing halogens and sulphur give lower numbers, averaging 0·2564.

The sequel will show that these numbers cannot be regarded as experimental deviations from a constant; but that they have significance can hardly be denied. It would also appear that the nature of the element in the compound, and its amount influence the absolute value; though the nature and extent of this influence can hardly be formulated.

The second law is:—*If the amounts of heat required to produce unit increase of volume in the passage from the liquid to the gaseous state be compared at different pressures for any two bodies, then the ratio of the amount at the boiling-point under a pressure* p, *to the amount at another pressure* p_2, *is approximately constant for all liquids.*

Or, $\frac{L}{s_1 - s_2}$ at p_1, bears a constant proportion to $\frac{L}{s_1 - s_2}$ at p_2, for all liquids, and probably for all solids.

* Determined by Horstmann, *Annalen*, 1868, Suppl. vi. p. 64.

† Determined by Ramsay and Young.

‡ Density taken as 39 (H=1) as an approximation.

In support of this statement the facts to be adduced are as follows :—

Water.					Chloroform.			
P. mms.	T.	s_1-s_2. ccs.	L.	$\frac{L}{s_1-s_2}$.	T.	s_1-s_2 ccs.	L.	$\frac{L}{s_1-s_2}$.
300	76°4	4022	552·1	0·137	35°0	535	64·65	0·121
760	100·0	1695	556·5	0·317	61·0	229	61·2	0·267
1000	107·1	1313	531·5	0·405	69 3	178	60·4	0·339
2000	129·3	695	515·6	0·742	92·5	95·3	58·2	0·611
3000	143 3	479	505·7	1·055	108·5	66 3	56·6	0·854
4000	154·2	369	497·7	1·350	121·0	51·4	55·4	1·079
5000	162·8	301	491·6	1 633	131·0	42 1	54·5	1·294
6000	170·1	255	486·1	1·906	140·0	35 9	53·6	1·493
7000	177·0	222	481·3	2·168	148·0	31·4	52·8	1·684
8000	182·9	197	477·0	2·424	155·0	27·9	52·2	1·871

The ratios, placing $\frac{L}{s_1-s_2}$ at 300 millims. pressure = 1·0 for each substance, are :—

P	300	760	1000	2000	3000	4000	5000	6000	7000	8000
Water	1·00	2·30	2·94	5·40	7·68	9·81	11·88	13·86	15·77	17·63
Chloroform	1·00	2·22	2·81	5·06	7·07	8·94	10·72	12·38	13·96	15·51

Another method of comparison is to regard the value of $\frac{L}{s_1-s_2}$ for water as equal to unity at each pressure, when the numbers for chloroform become

1·00 0·962 0·953 0·938 0·922 0·911 0·903 0·893 0·885 0·850.

Similar results may be obtained on comparing other bodies with water; it may be, however, remarked that the instance chosen is one of those in which least concordance is to be noticed. It is evident that here also the concordance is merely approximate.

Another point worthy of notice is this. The heat of vaporization is expended in at least two channels :—it produces expansion against pressure, thus doing external work; and it is partly expended in internal work on the molecules of the body. Now it follows from what has preceded that the internal and total work bear an approximately constant proportion to each other for any one pressure, whatever be the liquid. Thus at a pressure of approximately 760 millims. the ratio of external to total work is shown as follows :—

Substance.	Total work. External work.
Bromine	11·09
Phosphorus chloride	10·09
Carbon disulphide	10·39
Methyl alcohol	12·53
Ethyl alcohol	13·33
*Ethyl oxide (ether)	11·55
Water	13·02
Methyl iodide	10·45
Ethyl iodide	10·69
Methyl formate	11·55
Methyl acetate	12·51
Ethyl acetate	11·81
Ethyl oxalate	11·67
Amyl alcohol	13·32
Ethyl chloride	11·39
Benzene	10·25
Chloroform	10·99
Carbon tetrachloride	10·35
Mercury	12·36
*Nitric peroxide	12·42
Acetic acid	13·18
Water	13·02

* Vapour-density found at a few degrees above boiling-point.

In the equation $\frac{L}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{J}$, both terms are approximately known for a number of substances. But before comparing them it is necessary to reduce the pressures which have been given in the previous part of this paper in millimetres of mercury, to grams per square centimetre. The comparison has been made for water, with the following results :—

Pressures, in millim. of mercury.	$\frac{L}{s_1 - s_2}$.	$\frac{dp}{dt} \cdot \frac{t}{J}$.
300 . . .	0·137	0·1395
1000 . . .	0·405	0·414
2000 . . .	0·742	0·770
3000 . . .	1·055	1·110
5000 . . .	1·633	1·743

The non-equality of these numbers may be ascribed to various causes : want of accuracy in experiment, most probably in determination of the heats of vaporization, and the assumption that the density of the saturated vapour of water is normal at high pressures, are among the most likely. With such results for water, it was thought unnecessary to compare these terms for other liquids, where the constants are probably less accurately determined. These numbers, however, suffice to prove that the relations pointed out in the first part of this paper, although derived from observations and assumptions not always trustworthy, agree fairly well with those deduced from the vapour-pressures, which have been determined with a high degree of accuracy.

We now proceed to exhibit the relations between the vapour-pressures of various liquids. The constants of the bodies considered were determined by Regnault, Olszewski, Naumann, Isambert, Moitessier and Engel, and by ourselves.

If a curve be constructed to represent the relation of temperature to pressure for any substance, and if tangents be drawn to touch the curve at various points corresponding to certain temperatures, these tangents will give the rate of increase of pressure per unit rise of temperature ; in other words, the value $\frac{dp}{dt}$ for those temperatures.

If we construct curves for a number of substances, and determine the value of $\frac{dp}{dt}$ for each of them at the same temperature, it is clear that the values obtained will differ widely, and will be greater for volatile substances than for those which are less volatile. But if we determine the values of $\frac{dp}{dt}$ for the same series of bodies, not at the same temperature, but at the same pressure, the conditions under which the comparison is made will be more similar, and the resulting values may be expected to differ much less.

In the calculation of the vapour-pressures of a number of substances for each degree between certain limits of pressure, it became evident that at any given pressure the rate of increase was generally, though not always, greater for the volatile substances than for the less volatile.

It was found that the product of the absolute temperature into the rate of increase of pressure $\left(\frac{dp}{dt} \cdot t\right)$ at any given pressure was approximately the same for the bodies examined, but the differences were evidently too great to be ascribed to errors of experiment or of calculation. That this product should be approximately the same for different substances might perhaps be anticipated from the following considerations :—If there are two bodies, the absolute temperatures of which must be raised to 200° and 400° respectively, in order to produce a certain effect, the same for both, it might be expected that a further rise of temperature of 1° would produce a greater effect on the substance whose temperature was

200° than on that at 400°; for the rise of temperature in the first case is greater in proportion to the temperature to which the body has already attained than it is in the second, the rise in the one case being from 200° to 201°, and in the other from 400° to 401°. The rise of temperature would perhaps rather be proportional if the temperature of the hotter body were raised from 400° to 402°, or if the rise of temperature in each case were made proportional to the absolute temperature. In order, then, to make the conditions as similar as possible in the calculation of the value of $\frac{dp}{dt}$ for different substances, the magnitude of the unit degree of temperature should be made to vary in the same ratio as the absolute temperature of those substances, corresponding to the particular pressure at which they are compared; or, in other words, keeping the unit degree constant, the value $\frac{dp}{dt}$ should be multiplied by the absolute temperature t.

The values of $\frac{dp}{dt} \cdot t$ were determined for a number of liquids at several different pressures; and it was found that the products obtained for different stable substances at the same pressure were always approximately the same, whatever that pressure might be. Thus at a pressure of 400 millims. the following values were obtained:—

Carbon bisulphide . .	4436	1·000
Alcohol	5873	1·324
Chlorobenzene . . .	4724	1·065
Bromobenzene . . .	4703	1·060
Aniline	5124	1·155
Methyl salicylate . .	4959	1·112
Bromonaphthalene .	4930	1·111
Mercury	4812	1·085

This may be better seen by making one of these values equal to unity, and reducing the others in the same ratio. The second column is thus calculated.

Repeating this operation at another pressure, a second series of values was obtained; and these were reduced in the same

way. It was at once noticed that the reduced numbers at the new pressure were identical within the limits of error of experiment and calculation with those at the first pressure.

The calculations were therefore continued so as to include the widest attainable ranges of pressure, and at every pressure the value of $\frac{dp}{dt}.t$ for one substance was made equal to unity; and the others reduced in the same ratio. The values of $\frac{dp}{dt}.t$ for water were made $=1$, because the vapour-pressures of this substance have been determined by Regnault between wide limits of temperature with very great care; but it was noticed that the values for both water and alcohol were very much higher than for the other substances examined; and a similar comparison was therefore made by taking the values for carbon bisulphide as equal to unity.

It was then found that for pressures ranging between 150 and 1500 or 2000 millim. the reduced values for each substance, with the exception of mercury, were very nearly constant for all pressures.

In the case of alcohol and water, employing the vapour-pressures of water as determined by Regnault (*Mémoires de l'Académie*, vol. xxv.), and of alcohol as determined by Ramsay and Young, the ratio of the values $\frac{dp}{dt}.t$ was seen to be the same at pressures between 150 and 20,000 millim.

Two methods of calculation were adopted. In the first and more accurate method the vapour-pressures for each degree were calculated by the method of differences; and from these numbers the value of $\frac{dp}{dt}$ at any pressure could easily be calculated with very small error. The time required by this process is very great; and it was only adopted in the case of water, for which the pressures for each degree between $-32°$ and $230°$ have already been calculated (Balfour Stewart, 'Treatise on Heat'), and of carbon bisulphide, alcohol, chlorobenzene, bromobenzene, aniline, methyl salicylate, bromonaphthalene, and mercury between certain limits of pressure, generally 150 or 200 to 700 millim. (Ramsay and Young, Chem. Soc. Journ. 1885, p. 610).

In all other cases tangents were drawn to touch the vapour-pressure curves at points corresponding to definite pressures, several curves being required for each substance on different scales to admit of this being done. With care, this method yields fairly satisfactory results.

It has been mentioned that the identity of the reduced numbers for any one substance holds at pressures ranging between about 150 and 1500 to 2000 millim. At lower pressures it is much more difficult to determine the values of $\frac{dp}{dt}$ with accuracy, and the influence of experimental errors is greater. Moreover, it has been necessary provisionally to adopt the method of tangents for all substances except water and carbon bisulphide at low pressures; and it is doubtful, therefore, whether much confidence is to be placed in the values at pressures below 150 millim.

As regards high pressures, with the exception of alcohol all the substances have been investigated by Regnault; and it may be worth while to mention one or two facts which appear to throw some little doubt on the accuracy of some of his determinations at pressures above 2000 or 3000 millim. In vol. xxvi. of the *Mémoires de l'Académie* Regnault gives the vapour-pressures of a large number of substances for each 5°, calculated from his empirical formulæ. Many of these series have been examined by the method of differences; and it has been found that in several cases the numbers forming the third set of differences increase slowly up to pressures of from 2000 to 3000 millim., but decrease at higher pressures. This is notably the case with alcohol; and it has been shown by Ramsay and Young, in a paper read before the Royal Society in May 1885, that the higher vapour-pressures of this substance as determined by Regnault are too low. It has also been pointed out by Vincent and Chappuis (*Compt. Rend.* vol. c. p. 1216) that the vapour-pressures of methyl chloride above 7 atmospheres are much higher than those calculated from Regnault's formula, the difference at 140° amounting to no less than 11·3 atmospheres.

Where this decrease is observable in the third set of differences, it seems therefore justifiable to doubt the accuracy of the results at the higher pressures.

It may be stated in conclusion:—(1) That the values of $\frac{dp}{dt}.t$ are approximately the same for all stable substances at the same pressure, but that the differences are real, and are not due to errors of experiment or calculation; and (2) that the rate of increase of this value $\frac{dp}{dt}.t$ with rise of pressure is the same for all stable bodies, at any rate for pressures between 150 and 2000 millim., while for alcohol and water it is the same for all pressures between 150 and 20,000.

In the tables which follow, the values of $\frac{dp}{dt}$, t (absolute temperature), and $\frac{dp}{dt}.t$ are given for a number of stable substances at definite pressures, also the reduced values of $\frac{dp}{dt}.t$, that for (1) water and (2) carbon bisulphide being made $=1{\cdot}000$ at each pressure, the values for other substances at the same pressure being reduced in the same ratio.

The second series of tables contains similar data for several substances which dissociate more or less completely on their passage into the gaseous state. It will be seen that for such bodies the values of $\frac{dp}{dt}.t$ at any pressure are considerably higher than for stable substances at the same pressure.

The initial letter of the name of the observer of the vapour-pressure of each substance is given in the table containing the values of $\frac{dp}{dt}$ at the foot of the vertical columns. R. stands for Regnault, R. and Y. for Ramsay and Young, O for Olszewski, N. for Naumann, I. for Isambert, M. & E. for Moitessier and Engel.

Absolute Temperatures (t) of various Stable Substances corresponding to Definite Vapour-pressures.

Pressure in millims.	Carbon bisulphide. CS_2.	Ethyl iodide. C_2H_5I.	Ethyl bromide. C_2H_5Br.	Ethyl chloride. C_2H_5Cl.	Carbon tetrachloride. CCl_4	Ether. $C_4H_{10}O$.	Bromo-benzene. C_6H_5Br.	Chloro-benzene. C_6H_5Cl.	Chloroform. $CHCl_3$.	Mercury. Hg.	Ethylene. C_2H_4.	Sulphur. S.	Bromo-naphthalene. $C_{10}H_7Br$.	Methyl salicylate. $C_8H_8O_3$.	Aniline. $C_6H_5NH_2$.	Methyl alcohol. CH_3OH.	Water. H_2O.	Ethyl alcohol. C_2H_5OH.
10	.....	.....	.....	.....	.....	.....	.....	.....	.....	450·0	123·0	.....	.....	.....	.....	259·0	284·3	269·25
50	254·05	.....	250·0	.....	281·0	.....	347·3	.....	.....	506·9	139·5	.....	.....	405·5	375·0	288·0	311·8	295·0
100	267·7	291·0	262·8	.....	295·0	260·3	364·3	343·5	.....	584·3	146·75	.....	.....	424·3	392·9	296·0	329·7	307·5
150	276·5	300·0	271·0	248·7	304·0	268·5	374·9	353·9	.....	551·5	151·4	.....	485·9	436·4	404·0	303·0	383·1	315·3
200	283·2	307·0	277·0	254·3	311·0	274·8	383·2	361·8	208·0	564·3	154·9	.....	496·6	445·5	412·4	308·9	339·6	321·0
300	293·2	317·7	286·8	262·9	321·8	284·0	395·8	373·7	307·7	583·0	160·0	668·0	512·5	459·5	424·75	317·6	348·9	329·8
400	300·75	325·7	293·9	269·4	330·0	291·0	405·4	382·8	315·0	597·0	164·0	683·7	524·5	470·05	434·2	324·0	356·0	336·0
500	306·9	332·2	299·8	274·9	336·3	296·7	413·3	390·2	321·0	608·2	.....	696·1	534·1	478·7	441·9	329·3	361·7	341·1
600	312·15	.....	304·7	279·3	342·0	301·5	420·0	396·5	326·2	617·7	.....	707·0	542·3	486·0	448·5	333·9	366·5	345·45
700	316·75	.....	309·0	283·3	346·8	305·7	425·9	402·0	330·9	625·9	.....	716·1	549·5	492·4	454·25	337·8	370·7	349·1
800	320·8	.....	313·0	287·0	351·0	309·3	431·2	.....	334·8	633·1	.....	724·6	555·9	498·1	459·4	341·1	374·45	352·65
900	324·5	.....	316·3	290·1	355·0	312·9	.....	.....	338·3	639·8	.....	732·0	.....	.....	.....	344·1	377·8	355·6
1000	328·0	.....	319·6	293·0	359·0	316·0	.....	.....	341·7	645·4	.....	739·0	.....	.....	.....	343·1	380·85	358·4
1500	341·9	.....	332·8	305·0	373·9	328·5	.....	.....	355·1	669·2	.....	763·0	.....	.....	.....	358·9	393·2	369·8
2000	352·3	.....	342·7	314·3	385·3	338·0	.....	.....	365·5	.....	.....	788·2	.....	.....	.....	367·1	402·5	378·0
3000	368·7	.....	358·0	328·5	403·1	352·8	.....	.....	381·3	.....	.....	820·8	.....	.....	.....	380·1	416·5	390·6
5000	391·7	.....	379·3	348·1	427·9	373·4	.....	.....	404·2	.....	.....	.....	.....	.....	.....	398·1	435·85	408·3
10,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	465·65	435·7
15,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	485·3	453·5
20,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	500·5	467·1

Pressure in millims.	Ethyl alcohol. C_2H_5OH.	Water. H_2O.	Methyl alcohol. CH_3OH.	Aniline. $C_6H_5NH_2$.	Methyl salicylate. $C_8H_8O_3$.	Bromo-naphthalene. $C_{10}H_7Br$.	Sulphur. S.	Ethylene. C_2H_4.	Mercury. Hg.	Chloroform. $CHCl_3$.	Chloro-benzene. C_6H_5Cl.	Bromo-benzene. C_6H_5Br.	Ether. $C_4H_{10}O$.	Carbon tetrachloride. CCl_4.	Ethyl chloride. C_2H_5Cl.	Ethyl bromide. C_2H_5Br.	Ethyl iodide. C_2H_5I.	Carbon bisulphide. CS_2.	
10	0·68	0·65	0·738	.....	.....	.....	.....	1·43	0·302	.....	.....	.....	.....	.....	.....	.....	.....	.....	
50	2·89	2·70	3·04	2·00	1·93	.....	.....	4·54	1·31	.....	.....	2·13	.....	.....	.....	.....	.....	2·66	
100	5·35	4·95	5·48	3·77	3·49	4·12	.....	.....	2·47	.....	4·03	3·83	.....	2·57	.....	2·86	.....	4·78	R.
150	7·62	6·95	7·43	5·38	4·65	5·24	5·62	9·67	3·45	.....	5·60	5·19	5·01	4·52	.....	5·19	4·52	6·65	R.
200	9·83	8·88	9·54	6·76	6·05	7·38	7·12	12·41	4·40	8·64	7·07	6·59	7·18	6·30	7·64	7·15	6·28	8·38	R.
300	13·77	12·50	13·40	9·41	8·30	9·40	8·71	15·97	6·26	11·85	9·82	9·24	9·21	8·07	9·94	8·80	8·00	11·70	R.
400	17·48	15·90	17·16	11·80	10·55	11·35	9·97	.....	8·06	15·25	12·34	11·60	12·67	10·99	13·45	12·25	11·03	14·75	R.
500	21·10	19·10	20·56	14·10	12·65	13·10	11·33	22·67	9·78	18·16	14·68	13·85	16·33	13·88	16·94	15·47	14·08	17·65	R.
600	24·80	22·20	23·71	16·30	14·63	14·75	13·05	30·07	11·44	21·00	17·00	15·90	18·91	16·90	20·08	18·86	17·65	20·38	R.
700	28·20	25·30	27·41	18·40	16·50	16·35	14·60	.....	13·05	23·76	19·10	18·05	22·30	19·50	22·90	21·65	20·38	23·10	R.
800	31·46	28·10	30·63	20·40	18·40	18·40	16·60	.....	14·60	26·32	.....	19·95	25·45	21·47	26·91	24·39	23·10	25·67	R.
900	35·42	31·30	33·25	.....	.....	.....	18·40	.....	15·90	28·70	.....	.....	27·58	23·94	29·17	27·33	25·67	28·60	R.
1000	38·58	34·00	36·68	.....	.....	.....	20·40	.....	17·65	31·74	.....	.....	30·55	26·28	32·47	29·70	28·60	30·80	R.
1500	53·32	47·22	50·52	.....	.....	.....	28·10	.....	24·60	42·78	.....	.....	45·11	38·28	48·25	44·25	32·45	42·43	R.
2000	66·61	59·80	65·39	.....	.....	.....	35·35	.....	.....	52·98	.....	.....	58·54	48·75	60·90	55·00	75·10	52·40	R.
3000	92·09	83·40	88·57	.....	.....	.....	.....	.....	.....	72·94	.....	.....	78·00	65·00	83·80	75·10	108·9	70·67	R.
5000	138·28	125·00	134·24	.....	.....	.....	.....	.....	.....	104·70	.....	.....	116·00	96·62	119·80	108·9	.....	105·00	R.
10,000	225·40	215·30	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	R.
15,000	327·40	294·00	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	R.
20,000	408·40	365·00	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	R.
	R. (10 to 100 mm.) and R. & Y. (above 100 mm.).																		

dt for various Stable Substances at Definite Pressures.

Values of $\frac{dp}{dt} \cdot t$ (where t is the absolute temperature) for various Stable Substances at Definite Pressures.

Pressure in mms.	Carbon bisulphide. CS_2.	Ethyl iodide. C_2H_5I.	Ethyl bromide. C_2H_5Br.	Ethyl chloride. C_2H_5Cl.	Carbon tetrachloride. CCl_4.	Ether. $C_4H_{10}O$.	Bromobenzene. C_6H_5Br.	Chlorobenzene. C_6H_5Cl.	Chloroform. $CHCl_3$.	Mercury. Hg.	Ethylene. C_2H_4.	Sulphur. S.	Bromo-naphthalene. $C_{10}H_7Br$.	Methyl salicylate. $C_8H_8O_3$.	Aniline. $C_6H_5NH_2$.	Methyl alcohol. CH_3OH.	Water. H_2O.	Ethyl alcohol. C_2H_5OH.
10	.....	...	.....	.....	.....	.....	.....	.....	.....	136	176	.....	.....	.....	.....	191	1848	183
50	675·7	...	715	.....	722	.....	740	.....	.....	664	634	.....	.....	783	750	860	840·5	853
100	1260	1315	1364	.....	1333	1304	1395	1384	.....	1320	1419	.....	.....	1481	1481	1617	1607	1642
150	1839	1884	1938	1900	1915	1928	1946	1982	.....	1903	1879	.....	2002	2029	2174	2251	2315	2403
200	2373	2456	2438	2528	2510	2531	2525	2558	2575	2483	2474	.....	2692	2695	2788	2947	3016	3155
300	3490	3504	3513	3536	3537	3598	3657	3670	3646	3650	3627	3754	3782	3814	4014	4256	4361	4541
400	4436	4586	4547	4564	4580	4752	4703	4724	4772	4812	4920	4868	4930	4959	5124	5560	5660	5873
500	5417	...	5654	5520	5684	5612	5724	5728	5829	5918	.....	6063	6062	6056	6231	6770	6903	7197
600	6361	...	6597	6697	6669	6723	6678	6740	6850	7067	.....	7049	7104	7110	7311	7917	8136	8567
700	7317	...	7537	7624	7446	7673	7688	7678	7862	8168	.....	8113	8105	8125	8358	9259	9379	9845
800	8235	...	8554	8372	8403	8530	8602	.....	8812	9243	.....	9239	9089	9165	9372	10448	10522	11094
900	9281	...	9394	9420	9330	9559	.....	.....	9709	10173	.....	10210	.....	.....	.....	11441	11825	12595
1000	10102	...	10371	10290	10170	10513	.....	.....	10846	13391	.....	11208	.....	.....	.....	12585	12949	13827
1500	14507	...	14726	14716	14470	14819	.....	.....	15191	16463	.....	15733	.....	.....	.....	18131	18567	19718
2000	18469	...	18848	19141	18783	19787	.....	.....	19364	.....	.....	20706	.....	.....	.....	24004	24070	25179
3000	26056	...	26886	27528	26564	27730	.....	.....	27813	.....	.....	29015	.....	.....	.....	33666	34736	35970
5000	41128	...	41306	41702	41344	43814	.....	.....	42320	.....	.....	.....	.....	.....	.....	53440	54482	56460
10,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	100250	103870
15,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	142680	148480
20,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	182680	190770

Reduced Values of $\frac{dp}{dt} \cdot t$, that for Water being made $=1{\cdot}000$ at each pressure, and the values for other substances at the same pressure reduced in the same ratio.

Pressure in millims.	Carbon bisulphide. CS_2.	Ethyl iodide. C_2H_5I.	Ethyl bromide. C_2H_5Br.	Ethyl chloride. C_2H_5Cl.	Carbon tetrachloride. CCl_4.	Ether. $C_4H_{10}O$.	Bromobenzene. C_6H_5Br.	Chlorobenzene. C_6H_5Cl.	Chloroform. $CHCl_3$.	Mercury. Hg.	Ethylene C_2H_4.	Sulphur. S.	Bromonaphthalene. $C_{10}H_7Br$.	Methyl salicylate. $C_8H_8O_3$.	Aniline. $C_6H_5NH_2$.	Methyl alcohol. CH_3OH.	Water. H_2O.	Ethyl alcohol. C_2H_5OH.
10	.....	.....	.....	.....	.....	.....	.....	.....	.....	·736	·951	.....	.....	.....	.....	1·034	1·000	·991
50	·804	.....	·851	.....	·859	.....	·880	.....	.....	·790	·754	.....	.....	·931	·892	1·024	„	1·014
100	·796	·819	·849	.....	·830	·811	·868	·861	.....	·821	·863	.....	.. ...	921	·922	1·007	„	1·022
150	·794	·814	·837	·821	·827	·833	·840	·856	.....	·822	·812	.....	·865	·877	·939	·972	„	1·038
200	·787	·814	·808	·838	·832	·839	·837	·848	·854	·823	·820	.....	·863	·894	·924	·977	„	1·046
300	·787	·804	·806	·811	·811	·825	·839	·842	·836	·837	·832	·861	·867	·874	·916	·976	„	1·041
400	·784	·810	·803	·806	·809	·840	·831	·835	·843	·850	·869	·860	·871	·876	·905	·982	„	1·038
500	·784	.....	·818	·799	·823	·812	·829	·829	·844	·861	.....	·878	·877	·877	·902	·980	„	1·042
600	·782	.....	·811	·823	·819	·826	·821	·828	·842	·869	.....	·866	·873	·874	·899	·973	„	1·053
700	·780	.....	·804	·813	·794	·818	·820	·819	·838	·871	.....	·865	·864	·866	·891	·987	„	1·050
800	·783	.....	·813	·796	·799	·811	·818	.....	·838	·878	.....	·878	·864	·871	·891	·993	„	1·054
900	·785	.....	·794	·797	·789	·808	.....	.....	·821	·860	.....	·864	.....	.....	.....	·968	„	1·065
1000	·780	.....	·801	·795	·785	·812	.....	.....	·838	·880	.....	·865	... ..	.....	.....	·972	„	1·068
1500	·781	.....	·793	·793	·779	·798	.....	.. ...	·818	·887	.....	·847	.....	.....	.....	·977	„	1·062
2000	·767	.....	·785	·795	·780	·822	.....	.....	·805	.....	.....	·860	.....	.....	.....	·997	„	1·046
3000	·750	.....	·774	·793	·765	·798	.....	.....	·801	.....	.....	·836	.....	.....	.....	·969	„	1·036
5000	·755	.....	·758	·765	·759	·795	.....	.....	·777	.....	.....	.....	.....	.....	.....	·981	„	1·036
10,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	„	1·036
15,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	„	1·041
20,000	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	„	1·044

Reduced values of $\frac{dp}{dt} \cdot t$, that for Carbon-bisulphide being made $=1{\cdot}000$ at each pressure, and the values for other substances at the same pressure reduced in the same ratio.

Pressure in millims.	Carbon bisulphide. CS_2.	Ethyl iodide. C_2H_5I.	Ethyl bromide. C_2H_5Br.	Ethyl chloride. C_2H_5Cl.	Carbon tetrachloride. CCl_4.	Ether. $C_4H_{10}O$.	Bromo-benzene. C_6H_5Br.	Chloro-benzene. C_6H_5Cl.	Chloroform. $CHCl_3$.	Mercury. Hg.	Ethylene. C_2H_4.	Sulphur. S.	Bromo-naphthalene. $C_{10}H_7Br$.	Methyl salicylate. $C_8H_8O_3$.	Aniline. $C_6H_5NH_2$.	Methyl alcohol. CH_3OH.	Water. H_2O.	Ethyl alcohol. C_2H_5OH.
10	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
50	1·000	.....	1·058	.....	1·068	.....	1·095	.....	.....	0·983	·938	.....	.....	1·158	1·110	1·273	1·244	1·262
100	,,	1·028	1·066	.....	1·042	1·019	1·090	1·082	.....	1·031	1·109	.....	.....	1·157	1·157	1·264	1·256	1·283
150	,,	1·025	1·054	1·033	1·042	1·048	1·058	1·078	.....	1·035	1·022	.....	1·089	1·104	1·182	1·222	1·259	1·307
200	,,	1·035	1·027	1·065	1·058	1·066	1·064	1·078	1·085	1·046	1·043	.....	1·097	1·136	1·175	1·242	1·271	1·329
300	,,	1·022	1·024	1·031	1·031	1·049	1·066	1·070	1·063	1·064	1·057	1·094	1·103	1·112	1·165	1·241	1·271	1·324
400	,,	1·034	1·025	1·029	1·032	1·071	1·060	1·065	1·076	1·085	1·109	1·097	1·111	1·118	1·155	1·253	1·276	1·324
500	,,	.....	1·044	1·019	1·052	1·036	1·057	1·057	1·076	1·098	.....	1·119	1·119	1·118	1·150	1·250	1·275	1·329
600	,,	.....	1·037	1·053	1·048	1·057	1·050	1·060	1·077	1·111	.....	1·108	1·117	1·118	1·149	1·244	1·279	1·347
700	,,	.....	1·030	1·042	1·018	1·049	1·051	1·049	1·075	1·116	.....	1·109	1·108	1·110	1·142	1·265	1·282	1·345
800	,,	.....	1·039	1·017	1·020	1·036	1·045	.....	1·070	1·122	.....	1·122	1·104	1·113	1·138	1·269	1·278	1·347
900	,,	.....	1·012	1·015	1·005	1·030	.....	.....	1·046	1·096	.....	1·100	.....	.....	.....	1·233	1·274	1·357
1000	,,	.....	1·027	1·019	1·007	1·041	.....	.....	1·074	1·128	.....	1·109	.....	.....	.....	1·246	1·282	1·369
1500	,,	.....	1·015	1·014	0·997	1·045	.....	.....	1·047	1·135	.....	1·085	.....	.....	.....	1·250	1·280	1·359
2000	,,	.....	1·021	1·037	1·018	1·072	.....	.....	1·049	.....	.....	1·122	.....	.....	.....	1·300	1·304	1·364
3000	,,	.....	1·032	1·057	1·020	1·064	.....	.....	1·067	.....	.....	1·114	.....	.....	.....	1·292	1·333	1·380
5000	,,	.....	1·004	1·014	1·001	1·053	.....	.....	1·029	.....	.....	.....	.....	.....	.....	1·299	1·325	1·373
10,000																		
15,000																		
20,000																		

Values of $\frac{dp}{dt}$ for various Dissociable Substances at definite Pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4.	Ammonium chloride. NH_4Cl.	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3.CH\{^{OH}_{OCH_3}$.	Chloral ethyl-alcoholate. $CCl_3.CH\{^{OH}_{OC_2H_5}$.
10	...	...	0·99	.....	.....
50	...	1·91	3·75	2·97	2·94
100	5·2	3·63	7·02	5·46	5·47
150	8·1	4·98	10·13	8·26	7·91
200	11·3	6·26	12·71	10·23	10·17
300	15·6	8·39	18·57	14·33	14·08
400	20·47	10·74	23·48	19·30	18·06
500	24·27	13·25	...	.....	22·34
600	...	15·70			
	R. and Y.	R. and Y.	N., I., and M and E.	R. and Y.	R. and Y.

Absolute Temperatures (t) of various Dissociable Substances corresponding to definite Vapour-pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4.	Ammonium chloride. NH_4Cl.	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3.CH\{^{OH}_{OCH_3}$.	Chloral ethyl-alcoholate. $CCl_3.CH\{^{OH}_{OC_2H_5}$.
10	...	...	270·9	.....	.....
50	...	524·7	289·9	331·2	334·0
100	256·0	542·8	299·4	343·3	346·1
150	263·1	554·6	305·1	350·8	353·7
200	268·1	563·3	309·8	356·1	358·1
300	275·9	577·3	316·1	364·2	367·4
400	281·4	587·9	321·0	370·3	373·8
500	285·9	596·0	...	.....	378·7
600	...	603·0			

Products of $\frac{dp}{dt}$ into Absolute Temperatures $\left(\frac{dp}{dt}.t\right)$ for various Dissociable Substances at definite Pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4.	Ammonium chloride. NH_4Cl.	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3.CH\{^{OH}_{OCH_3}$.	Chloral ethyl-alcoholate. $CCl_3.CH\{^{OH}_{OC_2H_5}$.
10	...	...	268	.....	.....
50	...	1002	1087	984	983
100	1331	1970	2102	1866	1893
150	2131	2762	3091	2898	2798
200	3030	3526	3938	3635	3642
300	4304	4844	5870	5219	5173
400	5787	6314	7537	7147	6751
500	6939	7897	...	.....	8460
600	...	9467			

Reduced Values of $\left(\frac{dp}{dt} \cdot t\right)$; that for Water being made = 1·000 at each pressure, and the values for other substances at the same pressure reduced in the same ratio.

Pressure in millim.	Nitrogen peroxide. N_2O_4.	Ammonium chloride. NH_4Cl.	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3.CH\{^{OH}_{OCH_3}$.	Chloral ethyl-alcoholate. $CCl_3.CH\{^{OH}_{OC_2H_5}$
10	.....	.....	1·451	.....	.....
50	.....	1·193	1·293	1·170	1·170
100	0·828	1·226	1·308	1·161	1·178
150	0·921	1·193	1·335	1·252	1·209
200	1·004	1·169	1·306	1·205	1·208
300	0·987	1·110	1·346	1·197	1·186
400	1·018	1·116	1·332	1·263	1·193
500	1·004	1·143	.....	.....	1·225
600	.....	1·164			

Reduced Values of $\left(\frac{dp}{dt} \cdot t\right)$; that for Carbon-bisulphide being made = 1·000 at each pressure, and the values for other substances reduced in the same ratio.

Pressure in millim.	Nitrogen peroxide. N_2O_4.	Ammonium chloride. NH_4Cl.	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3.CH\{^{OH}_{OCH_3}$.	Chloral ethyl-alcoholate. $CCl_3.CH\{^{OH}_{OC_2H_5}$.
10	...	...	...	.....	.....
50	...	1·483	1·609	1·456	1·456
100	1·043	1·540	1·643	1·458	1·480
150	1·159	1·502	1·681	1·576	1·522
200	1·277	1·486	1·659	1·532	1·535
300	1·255	1·412	1·711	1·521	1·508
400	1·238	1·423	1·699	1·611	1·522
500	1·281	1·458	...	.....	1·562
600	...	1·488			

A relation may be observed between the two substances bromobenzene and chlorobenzene, and also between ethyl chloride, ethyl bromide, and ethyl iodide. The ratio of the absolute temperatures of the bodies in either group corresponding to any given vapour-pressure is a constant. Thus the ratio of the absolute temperature of bromobenzene to that of chlorobenzene when the vapour-pressure of both is 100 millim. $= \frac{364·3}{343·5}$, or 1·061; and at the other pressures up to 700 millim. the ratio remains absolutely constant at 1·059.

The ratio of the absolute temperatures of ethyl bromide and ethyl chloride has been determined for pressures between 150 and 5000 millim., and the numbers found vary only between 1·089 and 1·091. The data for ethyl iodide are much less complete; the comparison with ethyl chloride can only be made at pressures between 150 and 500 millim., and with ethyl bromide between 50 and 500 millim. In the first case the ratio of the absolute temperatures varies between 1·206 and 1·209, and in the second case between 1·105 and 1·108.